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SINGULAR MEASURES AND TENSOR ALGEBRAS

BY

SADAHIRO SAEKI

Let X and Y be two compact (Hausdorff) spaces, and let

$$V = V(X, Y) = C(X) \otimes C(Y)$$

be the tensor algebra over X and Y [8]. We denote by $V^{\sim}$ the space of all $f \in C(X \times Y)$ for which there exists a sequence (f_n) in V such that $f_n \to f$ uniformly and $\sup_n \|f_n\|_V < \infty$. Then $V^{\sim}$ forms a Banach algebra with norm $\|f\|_{V^{\sim}} = \inf \sup_n \|f_n\|_V$, where the infimum is taken over all sequences (f_n) as above (cf. [9] and [10]). The algebra $V^{\sim}$ is often called the tilde algebra associated with V. Notice that the natural imbedding of V into $V^{\sim}$ is an isometric homomorphism (cf. Theorem 4.5 of [5]).

For infinite compact spaces X and Y, C. C. Graham [1] constructs a function $f \in V^{\sim} \backslash V$ such that $f^n \in V$ for all $n \geq 2$. In the present note, we shall prove that a natural analog of Theorem 2.4 of [7] holds for V. Let r be a natural number and let E be a subset of $\mathbf{Z}^r_+$. As in [7], we shall say that E is *dominative* if (a) it contains all the unit vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ and (b) whenever $(m_j) \in \mathbf{Z}^r_+$, $(n_j) \in E$, and $m_j \leq n_j$ for all indices j, then $(m_j) \in E$.

THEOREM. *Let X and Y be two infinite compact spaces, and let E be a dominative subset of $\mathbf{Z}^r_+$. Then there exist functions $f_1, \ldots, f_r$ in $V^{\sim}$ such that*

(a) $f_1^{m_1} \cdots f_r^{m_r} \notin V$ *if* $(m_j) \in E \backslash \{0\}$,
(b) $f_1^{n_1} \cdots f_r^{n_r} \in V$ *if* $(n_j) \in \mathbf{Z}^r_+ \backslash E$.

In order to prove this, let Γ be a locally compact abelian group with dual G. We denote by $A(\Gamma) = M_a(G)^\wedge$ the Fourier algebra of Γ(cf. [3] and [4]).

LEMMA 1. *Let Γ be an infinite locally compact abelian group, let F be a finite dominative set in $\mathbf{Z}^r_+$, and let $\eta > 0$. Then there exist $f_1, \ldots, f_r \in A(\Gamma)$ such that*

(i) $\|f_j\|_{A(\Gamma)} < 3$ *and* $\|f_j\|_\infty < \eta$ *for all indices* j,
(ii) $\|f_1^{m_1} \cdots f_r^{m_r}\|_{A(\Gamma)} > 1$ *if* $(m_j) \in F \backslash \{0\}$,
(iii) $\|f_1^{n_1} \cdots f_r^{n_r}\|_{A(\Gamma)} < \eta$ *if* $(n_j) \in \mathbf{Z}^r_+ \backslash F$.

Proof. We may assume that $\eta < 1$. We first deal with the case where Γ is discrete or, equivalently, G is compact (and infinite). By Theorem 2.4 of [7], there exist probability measures $\mu_1, \ldots, \mu_r$ in $M(G)$ such that the measure

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