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ON THE BOUNDARY BEHAVIOR OF FUNCTIONS IN THE SPHERICAL DIRICHLET CLASS

BY

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A classical result of Fatou has the almost immediate consequence that a holomorphic injection of the unit disc, Δ, in the Riemann sphere, $\hat{\mathbf{C}}$, had radial limits almost everywhere on $\partial\Delta$. This theorem is quite striking since the cluster sets of such "schlicht" functions may be quite bizarre. Later Beurling [1] showed that radial limits exist except on a set of logarithmic (inner) capacity zero for the wider class of meromorphic functions in the spherical Dirichlet class. Tsuji [2] extended Beurling's argument to show that we have limits in any Stolz region at points in $\partial\Delta$, and the images of radial segments to the boundary are of finite spherical length. Tsuji's exceptional set is also of capacity zero.

Here we examine the question of the boundary behavior of normally convergent (i.e., uniformly convergent in the spherical metric on compact subsets of Δ) sequences of meromorphic functions in the spherical Dirichlet class. First we set the notation. Let $\| \quad \|$ be the L^2 norm with respect to Lebesgue measure $dx\,dy$ in Δ. If f is almost everywhere differentiable on Δ let

$$Tf = |f'|/(1 + |f|^2).$$

The spherical Dirichlet class D^* is the set of functions f which are meromorphic in Δ and satisfy $A[f] = \|Tf\|^2 < \infty$. $A[f]$ is the spherical area of the Riemann surface of f^{-1} spread over $\hat{\mathbf{C}}$. For $f \in D^*$, set

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

whenever the limit exists. For $B \subset \partial\Delta$, cap B is the logarithmic inner capacity.

We will prove the following two theorems.

THEOREM 1. *Let $f_n \in D^*$ and suppose $f_n \to f$ normally and*

$$\sum \|Tf_n - Tf\|^2 < \infty. \tag{1}$$

Then there exists a set $E \subset \partial\Delta$ with cap $E = 0$ *such that $f_n(e^{i\theta}) \to f(e^{i\theta})$ for all $e^{i\theta} \in \partial\Delta \backslash E$.*

THEOREM 2. *Let $f_n, f \in D^*$ and assume $f_n \to f$ normally and $A[f_n] \to A[f]$. Then there is a subsequence f_{n_k} and a set $E \subset \partial\Delta$ of capacity zero, so that*

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