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POSSIBLE BRAUER TREES

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1. Introduction

Let G be a finite group and let p be a prime. Let K be a finite extension field of $\mathbf{Q}_p$, the p-adic numbers, and let R be the ring of integers in K. Let B be a p-block of $R[G]$ with a cyclic defect group $D \neq \langle 1 \rangle$. Let $\tau = \tau(B)$ be the Brauer tree associated to B. We will say that τ *belongs* to G.

The object of this paper is to show that most trees do not occur as Brauer trees. This answers a question raised in [3]. The proof relies on the classification of finite simple groups. As far as I know not a single tree can be eliminated without using this classification. We first prove a result which reduces the question to the study of simple groups and then use the classification of simple groups to get information about trees belonging to simple groups.

Let τ be a tree and let P_0 be a vertex of τ. Let n be a natural number. Then $(\tau, P_0)^n$ is defined to be the union of n copies of τ with the vertices P_0 identified. (A more precise definition can be found in Section 2.) Observe that $\tau \simeq (\tau, P_0)^1$ for any vertex P_0 of τ.

Two trees α and β are *similar* if there exists a tree γ such that

$$\alpha \simeq (\gamma, P_0)^m \quad \text{and} \quad \beta \simeq (\gamma, P_0')^n$$

for integers m, n. It is easily seen that similarity is an equivalence relation. See Lemma 2.2.

THEOREM 1.1. *Let G be a finite group and let τ be a tree which belongs to G. Then there exists a simple group H which is involved in G and a group $\tilde{H}$ where*

$$\tilde{H} = H \quad \text{if } |H| = p,$$
$$\tilde{H}' = \tilde{H} \text{ and } \tilde{H}/\mathbf{Z}(\tilde{H}) \simeq H \quad \text{if } |H| \neq p,$$

such that τ is similar to a tree that belongs to $\tilde{H}$. If furthermore $\tau = (\gamma, P_0)^n$ for some $n > 1$ then P_0 is the exceptional vertex of τ if τ has an exceptional vertex.

In case G is p-solvable, $|H| = p$. Thus Theorem 1.1 asserts the well-known

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