ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 2, Summer 1998

LOCAL PROPERTIES OF SELF-SIMILAR MEASURES

MANAV DAS

1. Introduction

Let (X, d) be a complete, separable metric space and let μ be a Borel probability measure on X. Let $B_\epsilon(x)$ be the closed ball of radius ϵ centered at x. For $x \in X$, $\alpha \geq 0$ we are interested in the quantity

$$\alpha = \lim_{\epsilon \to 0} \frac{\log \mu B_\epsilon(x)}{\log \mathrm{diam B}_\epsilon(\mathrm{x})},$$

if the limit exists. α is often referred to as the local dimension of μ at x. Typical questions involve the conditions on the measure that would ensure the existence of the limit, characterization of points for which this limit exists, the range of possible values for the local dimension and so forth. Several authors investigated this phenomenon using thermodynamic formalism. Cawley and Mauldin [3] were the first to provide a geometric measure-theoretical framework for such a decomposition for Moran fractals. Attention has primarily focussed on the situation where X is taken to be d-dimensional euclidean space, and μ is a self-similar measure with totally disconnected support. It is therefore natural to ask if such a decomposition may be carried out for these measures by weakening the condition that their support be totally disconnected, but requiring that they satisfy the open set condition (OSC). This problem was posed in [7, Question (d)] and [9, Question 7.8], among others, and was finally settled by Arbeiter and Patzschke [1], for a random self-similar measure.

We present an alternate treatment of the above questions. We restrict our attention to the case where $X = \mathbb{R}^d$ and μ is a graph directed self-similar measure satisfying the open set condition. Our setting is therefore more restrictive than the random setting in [1], but more general than the class of self-similar measures. Moreover our approach yields stronger results, and may be readily generalized to a broader class of measures that includes the ones stated above. We are able to establish some explicit local properties, which enable us to see the geometric measure-theoretic interplay between the sets that are used to construct the measure (called cylinders), and the sets used to study the local geometry (the balls). This approach allows us to transfer results about Hausdorff [4] and Packing dimensions [5] from the string space (which is easy to analyze) to the metric space in question. The paper is arranged as follows: Section 2 describes the setting. Section 3 begins with the notion of *stoppings*. We

Received April 28, 1997.
1991 Mathematics Subject Classification. Primary 28A75; Secondary 28A80.