ILLINOIS JOURNAL OF MATHEMATICS
Volume 32, Number 3, Fall 1988

MINIMAL ASYMPTOTIC BASES WITH PRESCRIBED DENSITIES

BY

PAUL ERDŐS AND MELVYN B. NATHANSON[1]

Dedicated to the memory of Irving Reiner

Let $h \geq 2$. The set A of integers is an *asymptotic basis of order h* if every sufficiently large integer can be represented as the sum of h elements of A. If A is an asymptotic basis of order h such that no proper subset of A is an asymptotic basis of order h, then the asymptotic basis A is *minimal*. It follows that if A is minimal, then for every element $a \in A$ there must be infinitely many positive integers n, each of whose representations as a sum of h elements of A includes the number a as a summand. Stöhr [6] introduced the concept of minimal asymptotic basis, and Härtter [2] proved that minimal asymptotic bases of order h exist for all $h \geq 2$. Erdös and Nathanson [1] have reviewed recent progress in the study of minimal asymptotic bases.

For any set A of integers, the *counting function* of A, denoted $A(x)$, is defined by $A(x) = \operatorname{card}(\{a \in A \mid 1 \leq a \leq x\})$. If A is an asymptotic basis of order h, then $A(x) > c_1 x^{1/h}$ for some constant $c_1 > 0$ and all x sufficiently large. For every $h \geq 2$, Nathanson [3], [4] has constructed minimal asymptotic bases that are "thin" in the sense that $A(x) < c_2 x^{1/h}$ for some $c_2 > 0$ and all x sufficiently large.

Let A be a set of integers. The *lower asymptotic density* of A, denoted $d_L(A)$, is defined by $d_L(A) = \liminf_{x \to \infty} A(x)/x$. If $\alpha = \lim_{x \to \infty} A(x)/x$ exists, then α is called the *asymptotic density* of A, and denoted $d(A)$. Nathanson and Sárközy [5] proved that if A is a minimal asymptotic basis of order h, then $d_L(A) \leq 1/h$. In this paper we construct for each $h \geq 2$ a class of minimal asymptotic bases A of order h with $d(A) = 1/h$. This result is best possible in the sense that it gives the "fattest" examples of minimal asymptotic bases. We also prove that for every $\alpha \in (0, 1/(2h-2))$ there exists a minimal asymptotic basis A of order h with $d(A) = \alpha$.

Received January 4, 1988.

[1]The research of the second author was supported in part by a grant from the PSC-CUNY Research Award Program of the City University of New York.