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LOW DIMENSIONAL SECTIONS OF BASIC SEMIALGEBRAIC SETS[1]

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To the memory of M. Raimondo

Introduction

Let $X \subset \mathbb{R}^n$ be a real algebraic set, and let $\mathscr{P}(X)$ denote the ring of polynomial functions on X. Recall that a subset $S \subset X$ is called *semialgebraic* if there exist polynomials $f_{ij}, g_i \in \mathscr{P}(X)$ such that

$$S = \bigcup_{i=1}^{p} \{x \in X: f_{i1}(x) > 0, g_i(x) = 0\}.$$

As is well known, if S is open the g_i's in this expression can be omitted. Recall also that an open semialgebraic set is called *basic open* if furthermore $p = 1$. These basic open sets have attracted a lot of interest in recent times, till the proof of the beautiful theorem that states that a basic open set S has always a description

$$S = \{x \in X: f_1(x) > 0, \dots, f_s(x) > 0\}$$

with $s \leq \dim(x)$; see [Br2, 3, 4], [Sch], [Mh], [AnBrRz1]. However, the problem of understanding when a given semialgebraic set is basic open and, in that case, how many inequalities are needed to generate it, is far from solved. An immediate remark is that if S is basic open, then $S \cap \overline{(\overline{S} \setminus S)}^Z = \emptyset$, where $\overline{}$ stands for the euclidean closure, and $\overline{}^Z$ for the Zariski closure. The only full characterization available is due to Bröcker and Scheiderer. To state it properly, let us say that a semialgebraic set S is *s-basic* if there are s polynomials $f_1, \dots, f_s \in \mathscr{P}(X)$ such that $S = \{f_1 > 0, \dots, f_s > 0\}$, and that S is *generically s-basic* if it is s-basic up to codimension 1, that is, there are s polynomials $f_1, \dots, f_s \in \mathscr{P}(X)$ and a nowhere dense algebraic subset $Z \subset X$ such that

$$S \setminus Z = \{f_1 > 0, \dots, f_s > 0\} \setminus Z.$$

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