ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 3, Fall 1994

DEFINING FRACTALS IN A PROBABILITY SPACE

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1. Introduction

In [1] Billingsley defined Hausdorff dimension for subsets of a probability space, and developed relations with entropy and information theory. In [2] he explored the effect of varying the probability measure in the definition. In revisiting his work we will sharpen his density theorems so that they give results for measures as well as dimension (Lemmas 5.1 and 5.2).

In Euclidean space $\mathbf{R}^d$ there is no generally accepted definition of a fractal, even though fractal sets are widely used as models for many physical phenomena. The idea behind these models is that of self-similarity or affineness which is based on the linear structure of $\mathbf{R}^d$. These and other geometrical notions have no obvious meaning in an abstract probability space. One of us [11] proposed a measure-theoretic definition for subsets $E \subset \mathbf{R}^d$: E should be called a fractal if

$$\dim(E) = \operatorname{Dim}(E), \tag{1.1}$$

where $\dim(E)$ is the familiar Hausdorff dimension and $\operatorname{Dim}(E)$ denotes packing dimension as defined in [10], using efficient packing by disjoint balls with center in E.

The first objective of this paper is to define packing measure and dimension in a probability space $(\Omega, \mathscr{F}, \mu)$. This is done in two stages: In Section 3 we produce a premeasure and obtain the analogue $\Delta(E)$ of the upper Minkowski index in Ω, while Section 4 completes the definition of packing measure and dimension. We can then use (1.1) as the definition of a fractal set in $(\Omega, \mathscr{F}, \mu)$ with respect to μ. In Section 5 we complete the development of density arguments relevant to both Hausdorff and packing measures and use these in Section 6 to provide (Corollary 6.4) a useful criterion for $A \subset \Omega$ to be a fractal of dimension c. In Section 7 we use ideas of Cutler [4] to analyse a measure ν on $(\Omega, \mathscr{F})$ with respect to μ. This leads to the analogue for probability spaces of the results obtained in [12] for $\mathbf{R}^d$.

Received July 13, 1992.
1991 Mathematics Subject Classification. Primary 60A10; Secondary 28A80.

[1]During the preparation of this paper, the author was a visiting scholar at the University of Virginia.

[2]Research partially supported by a grant from the National Science Foundation.