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APPROXIMATE VERSIONS OF CAUCHY'S FUNCTIONAL EQUATION

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1. Introduction

Ulam [U, page 63] raised the general problem of when a mathematical entity which nearly meets certain requirements must be close, in some sense, to one which does meet the requirements. A particular case is a result of Hyers [H]: if

$$|f(x+y) - f(x) - f(y)| < \varepsilon \quad \text{for all } x, y,$$

then there is a g satisfying Cauchy's equation with $|f(x) - g(x)| < \varepsilon$ for all x. A survey of related results appears in [HR].

In this note, we look at stronger assumptions ([H] did not even assume f was measurable) that imply $f(x) = \gamma x$ almost everywhere (we will use Lebesgue measure, denoted by μ, throughout). Our main results are:

THEOREM 1. *Let f, a, b be measurable functions and let*

$$\delta(x,y) \equiv f(x+y) - a(x) - b(y). \tag{1}$$

If there is a $J \in \mathbf{R}$ such that, for every $\varepsilon > 0$,

$$\mu(\{(x,y) | \, |\delta(x,y) - J| \geq \varepsilon\}) \tag{2}$$

is finite, then, for some γ and β, $f(x) = \gamma x + \beta$ almost everywhere.

Remarks

1. It is easy to see that, if $f = a = b$ and $J = 0$, then $\beta = 0$.

The referee points out that the case $f = a = b$ and $J \neq 0$ is related to Pexider's equation $f(x+y) = f(x) + f(y) + K$.

2. For any $p > 0$, $\delta \in L^p(R^2)$ implies that δ satisfies (2) with $J = 0$.

3. It can also be shown that, for some β', γ', $a(x) = \gamma' x + \beta'$ almost everywhere (the same argument applies to $b(x)$ by symmetry): replace f by

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