ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

ON A SPECIAL SCHAUDER BASIS FOR THE SOBOLEV SPACES $W_0^{1,p}(\Omega)$

HAMID BELLOUT

1. Introduction

We are interested in Schauder bases for the Sobolev spaces $W_0^{1,p}(\Omega)$ for Ω a general smooth subdomain of R^n. In the particular case where Ω is a cube it has been proved by Z. Ciesielski and J. Domsta in [3] that $W^{1,p}(\Omega)$ has a Schauder basis made of functions that are mutually orthogonal in $L^2(\Omega)$. Also, for the particular case $p = 2$ it is well known that the eigenfunctions of the Laplace operator constitute a basis of $W_0^{1,2}$ with the property that the elements of the basis are mutually orthogonal in L^2. For a general domain Ω and general p, the existence of a Schauder basis for $W_0^{1,p}(\Omega)$ was proved by S. Fucik, O. John and J. Necas in [4]. However, it is not known whether the elements of this basis are mutually orthogonal in L^2. The existence of a Schauder basis for $W_0^{1,p}(\Omega)$, for general p and Ω, made of elements that are mutually orthogonal in $L^2(\Omega)$ seem to be an open question. It should be mentioned that the Gram-Schmidt orthonormalization of a Schauder basis may fail to be a Schauder basis (see [11], [9]).

In this paper we prove the existence of a Schauder basis of $W_0^{1,p}(\Omega)$ with a property that is weaker than that of having elements that are mutually orthogonal in $L^2(\Omega)$ but that can usefully be substituted for it in some contexts. To wit, let $\{w^i\}_{i\geq 1}$ be a Schauder basis of $W_0^{1,p}(\Omega)$ and V_n the closure in L^2 of the subspace of $W_0^{1,p}(\Omega)$ spanned by the subsequence $\{w^i\}_{i>n}$. We show the existence of a basis with the property that

$$\forall n, \quad V_n \cap \operatorname{Span}\{w^1, \dots, w^n\} = 0. \tag{1}$$

Note that whenever the elements of a basis are mutually orthogonal in $L^2(\Omega)$, (1) is trivially satisfied.

Our interest in this problem came from needs encountered while dealing with the question of existence of a solution to a partial differential equation (see [2]). We elucidate the connection between the two: Schauder bases of Banach spaces are used in the theory of partial differential equations in connection with the Galerkin method. This is one of the methods commonly used to establish the existence of a solution to a partial differential equation.

Received November 20, 1992.
1991 Mathematics Subject Classification. Primary 46; Secondary 35.