ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 2, Summer 1995

DISTRIBUTION OF FUNCTIONS IN ABSTRACT H^1

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I. Introduction

Let A be a *weak*-Dirichlet algebra* i.e. a subalgebra A of $L^\infty(\mu)$ where $(\mathcal{M}, \mu)$ is a probability space such that: μ is multiplicative on A, A contains the constants and $A + \bar{A}$ is weak*-dense in $L^\infty(\mu)$.

The *abstract Hardy spaces* are defined by the following:

$\mathscr{H}^p(\mathcal{M})$ is the closure of A in $L^p(\mu)$, for $1 \leq p < \infty$,
$\mathscr{H}^\infty(\mathcal{M})$ is the weak*-closure of A in $L^\infty(\mu)$.

We also denote by $\mathscr{H}_0^1(\mathcal{M})$ the set of functions in $\mathscr{H}^1(\mathcal{M})$ with $\int_{\mathcal{M}} f d\mu = 0$ and by $\operatorname{Re}\mathscr{H}^1(\mathcal{M})$ the set of real parts of functions in $\mathscr{H}^1(\mathcal{M})$.

These algebras were introduced in [SW], where it was proven that the corresponding abstract Hardy spaces enjoy most of the measure theoretic properties of the original Hardy spaces. Then in [HR] the conjugate function was studied for these weak*-Dirichlet algebras. The conjugation operator is defined for $1 < p < \infty$ by

$$\mathscr{H}: L^p(\mu) \to L^p(\mu)$$

$$f \mapsto \tilde{f} \text{ such that } f + i\tilde{f} \in \mathscr{H}^p(\mathcal{M}) \text{ and } \int_{\mathcal{M}} \tilde{f} d\mu = 0.$$

This operator is bounded on $L^p(\mu), 1 < p < \infty$. For $p = 1$, $\mathscr{H}$ is only bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. So a natural question is to characterize the functions in $L^1(\mu)$ for which $\tilde{f}$ is in $L^1(\mu)$. This is the problem we will investigate here. Note that if $f \geq 0$, Zygmund's theorem (which holds for weak*-Dirichlet algebras, see [HR]) asserts that the condition for $\tilde{f}$ to be in $L^1(\mu)$ is that f is in $L \log_+ L$ (i.e., $\int_{\mathcal{M}} |f| \log_+(|f|)\, d\mu < \infty$).

We will first recall the solution of the problem for the classical Hardy spaces. It was solved on $\mathbf{T}, \mathbb{R}$ and $\mathbb{R}^n$ by B. Davis [Da], here is his result for $H^1(\mathbb{R})$. For f a real valued function on $\mathbb{R}$, let f_δ be the signed decreasing function (i.e., non-positive and not increasing on $(-\infty, 0)$, non-negative and not increasing on $(0; \infty)$) which has the same distribution as f and let $M(t) = \int_{-t}^{t} f_\delta(u)\, du$.

Received November 2, 1992
1991 Mathematics Subject Classification. Primary 46J10; Secondary 43A17.