A NORM INEQUALITY FOR A "FINITE-SECTION" WIENER-HOPF EQUATION

BY
GLEN BAXTER[1]

1. Introduction

We are concerned here with establishing a norm inequality for an equation which arises in a variety of interesting problems. This seemingly simple inequality has a surprisingly large number of applications which we have brought to the reader's attention in §3.

The result concerns the equation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)f(\theta)e^{-ik\theta}\,d\theta = g_k \qquad (0 \leqq k \leqq n), \tag{1.1}$$

where $f(\theta)$ is a sufficiently nice function, and where $h(\theta)$ is a polynomial of degree n in $e^{i\theta}$. The purpose is to relate the "size" of $h(\theta)$ to the "size" of $g(\theta) = \sum_{k=0}^{n} g_k e^{ik\theta}$. In particular, we find a norm inequality

$$\| h \| \leqq M \| g \|, \tag{1.2}$$

where $\| \cdot \|$ denotes the sum of the absolute values of the coefficients in the polynomials $h(\theta)$ and $g(\theta)$, and where the constant M is independent of the particular $g(\theta)$ and $h(\theta)$ involved. Such an inequality allows one to consider the convergence of a sequence of h's in terms of the corresponding sequence of g's.

Before stating the main result, let us generalize the norm used. Let $\nu(n) \geqq 1$ be a function of the integer n such that $\nu(n) \leqq \nu(m)\nu(n-m)$ for every n, m. Denote by $\mathfrak{A}_\nu$ the class of functions $F(\theta)$ integrable over $-\pi \leqq \theta \leqq \pi$ with Fourier coefficients F_k such that

$$\| F \|_\nu \equiv \sum_{-\infty}^{\infty} \nu(n) \mid F_n \mid < \infty. \tag{1.3}$$

Next, let us restrict the class of functions $f(\theta)$ considered in (1.1). Let $f(\theta)$ be integrable over $-\pi \leqq \theta \leqq \pi$ with Fourier coefficients c_k, let $D_n(f) = \det(c_{i-j})$ $(i, j = 0, 1, \cdots, n)$, and let $f(\theta)$ satisfy $\log f(\theta) \in \mathfrak{A}_\nu$. In terms of the notation just introduced, equation (1.1) can be written

$$\begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & c_0 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \end{pmatrix}. \tag{1.4}$$

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