COHOMOLOGY OF LIE GROUPS

Dedicated to Reinhold Baer on the occasion of his sixtieth birthday

BY

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1. Introduction

Given a Lie group G and a finite-dimensional continuous G-module V, there present themselves several kinds of "cohomology groups" for G in V that have representation-theoretical, topological, or group-structural interest. In a concrete, but conceptually unsatisfactory fashion, they may be described as the cohomology groups based on continuous cochains, differentiable cochains, or representative cochains. If G is a real linear algebraic group, there is a further specialization of cochains to rational representative cochains.

In order to bring these various cohomology theories under control, one must define and analyze the underlying categories of G-modules, as well as the appropriate notions of "resolution" of a G-module, capable of yielding technically efficient definitions of the cohomology groups. The key for obtaining satisfactory functorial definitions of cohomology groups of the type considered here is the notion of "injectivity" of a module, which (in the context of the general theory of modules) was considered and analyzed first by Reinhold Baer (Bull. Amer. Math. Soc., vol. 46 (1940), pp. 800–806) who showed, in particular, that every module can be imbedded in an injective module, thus ensuring the existence of injective resolutions. It turns out that, contrary to what is the case for projective resolutions, the mechanism of injective resolutions can be adapted to take account of additional structure (topological, differentiable, or rational).

For algebraic linear groups over arbitrary fields of characteristic 0, such a theory has been presented in [3], where it has also been shown how the rational cohomology of such a group can be expressed in terms of the usual cohomology of Lie algebras. The exactly analogous development for the representative cohomology of a Lie group is included below. The continuous cohomology theory has been presented in [11], which also contains the main results on the passage to Lie algebra cohomology. Examination of the technicalities involved in this passage has revealed difficulties which were not fully appreciated at the time when [11] was written, and which stand in the way of a truly categorical treatment.

These difficulties reside in the requirements of "differentiability" and "integrability" of a G-module, the first of which is the essential link to the Lie algebra cohomology, while the second is an indispensable technical aid. While it is immediate from classical results that requirements of this type

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