THE SIGN OF THE GAUSSIAN SUM

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY
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It is well known and easily proved that if p is an odd prime, then

$$(1) \qquad \sum_{s=0}^{p-1} e^{2\pi i s^2/p} = c\sqrt{p},$$

where $c = \pm 1$ if $p \equiv 1 \pmod 4$ and $c = \pm i$ if $p \equiv 3 \pmod 4$. (See, for example, the remark on Theorem 212 in Landau's *Vorlesungen über Zahlentheorie.*) As Gauss noted many years ago, it is a much more difficult matter to show that the plus sign must be taken in both cases. A proof originating from Kronecker is given by Hasse in his *Vorlesungen über Zahlentheorie* (pp. 449–452). It may be worthwhile to give a proof not very dissimilar from this but perhaps a trifle simpler and more self-contained.

Write

$$\zeta = e^{2\pi i/p}, \qquad P = \zeta - 1.$$

Then from the identity

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = \prod_{n=1}^{p-1} (x - \zeta^n)$$

and the equalities

$$(1 - \zeta^n)/(1 - \zeta) = 1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1},$$
$$(1 - \zeta)/(1 - \zeta^n) = 1 + \zeta^n + \zeta^{2n} + \cdots + \zeta^{(m-1)n},$$

where $mn \equiv 1 \pmod p$, we have

$$(2) \qquad p = \prod_{n=1}^{p-1} (1 - \zeta^n) = \varepsilon P^{p-1},$$

where ε is a unit. We prove further that

$$(3) \qquad \sqrt{p} = \prod_{n=1}^{(p-1)/2} \{2 \sin (2n\pi/p)\}.$$

In fact from the identity

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = \prod_{n=1}^{(p-1)/2} (x - \zeta^{2n})(x - \zeta^{-2n}),$$

we have

$$\begin{aligned} p &= \prod_{n=1}^{(p-1)/2} (1 - \zeta^{2n})(1 - \zeta^{-2n}) \\ &= \prod_{n=1}^{(p-1)/2} (\zeta^{-n} - \zeta^{n})(\zeta^{n} - \zeta^{-n}) \\ &= \prod_{n=1}^{(p-1)/2} \{2 \sin (2n\pi/p)\}^2, \end{aligned}$$

from which (3) follows, since each sine is positive.

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