ON THE CLASSIFICATION OF FILTERED MODULES

BY
Alex Heller[1]

1. Introduction

Throughout this paper Λ will denote an algebra with unit over a field K. Modules are all left Λ-modules; vector spaces are vector spaces over K. Graded vector spaces or modules are graded by nonnegative degrees, i.e., the homogeneous components of negative degrees are 0.

We propose to study filtered modules, for which we adopt the following definition as leading to an economy of notation. A *filtered module* is a short exact sequence

$$\boldsymbol{X} = (0 \longrightarrow X \xrightarrow{\xi} X \xrightarrow{\xi''} X'' \longrightarrow 0)$$

of graded modules, the maps ξ, ξ'' being homogeneous of degrees 1, 0. This is easily seen to coincide with the more familiar notion when the maps $\xi_q : X_q \rightarrow X_{q+1}$ are interpreted as inclusions.

If also $\boldsymbol{Y} = (0 \rightarrow Y \xrightarrow{\eta} Y \xrightarrow{\eta''} Y'' \rightarrow 0)$ is a filtered module, a map $\boldsymbol{\phi} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ is a pair of maps $\phi : X \rightarrow Y$, $\phi'' : X'' \rightarrow Y''$ such that $\phi\xi = \eta\phi$, $\phi''\xi'' = \eta''\phi$. These $\boldsymbol{\phi}$ constitute, in an obvious way, a graded category F which may be given an abelian structure [1]; we shall not however make any use of this structure here. In fact F may be interpreted as a "filtered category": this notion, which will not be investigated here, the reader may supply for himself.

The functor $S''\boldsymbol{X} = X''$, $S''\boldsymbol{\phi} = \phi''$ is of course homogeneous additive. Its value on $\boldsymbol{X}$ is the *associated graded module* of $\boldsymbol{X}$; we shall also refer to $\boldsymbol{X}$ as an *extension* of $S''X$.

Two extensions of a graded module A are *equivalent* if there is an equivalence $\boldsymbol{\phi}$ of filtered modules connecting them, such that $S''\boldsymbol{\phi} = 1 : A$. The problem to which we address ourselves here is that of classifying the equivalence classes of extensions of a fixed module A. The analogous problem for abelian groups has been treated by Shih [2], who arrives at a formulation not readily comparable with that given below. He also announces (but does not state) results for the case considered here.

In the very simple case that A has only two nonvanishing homogeneous components, say A_0 and A_1, the classification, viz. $\mathrm{Ext}^1(A_1, A_0)$, is of course well known. We shall see that a similar description is also valid in the general case. In the course of the discussion we shall also solve another problem, namely that of attaching to a filtered module a strong enough

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