ON THE INTEGRAL REPRESENTATION OF THE RATE OF TRANSMISSION OF A STATIONARY CHANNEL

BY

K. R. PARTHASARATHY

1. Introduction

For finite-memory channels as defined in [1] the equality of stationary and ergodic capacities has been proved by I. P. Tsaregradsky [2] and L. Breiman [3]. For channels with infinite memory we give a new definition of the rate of transmission which coincides with the usual definition for finite-memory channels. By utilising the representation of a general stationary measure as a direct integral of ergodic measures due to Kryloff and Bogoliouboff [4], we obtain a representation for the rate of transmission of any stationary input in terms of the rates for ergodic inputs. This representation leads to two important results: It shows that for any stationary channel the ergodic capacity is equal to the stationary capacity, and that the ergodic capacity is attained whenever the stationary capacity is attained.

2. Basic properties of stationary and ergodic inputs

In this section we shall consider stationary measures on the Borel field generated by cylinder sets of the product space

$$A^I = \prod_{i=-\infty}^{+\infty} A_i, \qquad A_i = A \quad \text{for all } i,$$

where the product is taken over all integers and A is a finite set consisting of a elements. Then under the product topology we can assume A^I to be a compact metric space. If T is the shift transformation of A^I into itself, then, under the group of automorphisms $T^n: n = \cdots, -1, 0, 1, \cdots$, A^I becomes a compact dynamical system. Hereafter we shall follow the notation and terminology of Oxtoby [5]. If $f(p)$ is a real-valued function on A^I, let

$$M(f, p, k) = f_k(p) = (1/k) \sum_{i=1}^{k} f(T^i p) \quad (k = 1, 2 \cdots) \tag{2.1}$$

and

$$M(f, p) = f^*(p) = \lim_{k \to \infty} M(f, p, k) \tag{2.2}$$

in case this limit exists. A Borel subset E of A^I is said to have invariant measure one if $\mu(E) = 1$ for every invariant probability measure μ. Let Q be the set of points p for which $M(f, p)$ exists for every $f \in C(A^I)$ where $C(A^I)$ is the space of continuous functions on A^I. It follows easily from Riesz's representation theorem that corresponding to any point $p \in Q$ there exists a unique invariant probability measure μ_p such that

$$M(f, p) = \int f \, d\mu_p .$$

Received July 16, 1960.