TORSION-FREE AND MIXED ABELIAN GROUPS

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I. Introduction

In the initial sections of this paper we classify arbitrary torsion-free abelian groups in a manner similar to that used to classify the subgroups of the rationals. An existence theorem is presented to complete the picture, and these results are applied to give new examples of indecomposable groups of any finite rank.

The remaining sections are concerned with certain countable mixed groups of torsion-free rank 1; they are essentially classified by the invariants which came up in Kaplansky and Mackey's solution [6] of the analogous problem for modules over complete discrete valuation rings. The proof of the present classification theorem depends heavily on the author's adaptation [10] of their work to modules over (not necessarily complete) discrete valuation rings. Again an existence theorem shows the set of invariants is complete. Although these invariants are clumsy, they are used to solve cancellation, square-root, and isomorphic refinement problems.

II. Basic definitions and notation

All groups considered are abelian and are written additively. If G is a group, the set of all elements in G of finite order forms a subgroup T, the *torsion subgroup* of G. G is *torsion* if $T = G$; G is *torsion-free* if $T = 0$. An arbitrary group may contain elements of infinite order and elements of finite order. Since most work on abelian groups has been done on torsion groups and torsion-free groups, a general group is called *mixed* to distinguish it from these particular cases.

Let p be a prime integer, x an element of G. x is *divisible by* p^n in case there is a $y \in G$ such that $p^n y = x$. x has *p-height* n, denoted $h_p(x) = n$, if n is maximal with the property that x is divisible by p^n; if there is no such n, x has *infinite p-height.*

Let I denote the rational integers, Q the rational numbers. If G is a group, $Q \otimes_I G$ is a vector space over Q. The *rank* of G is the dimension of $Q \otimes G$. (There are other notions of rank in abelian group theory; the one defined above is sometimes called the "torsion-free rank" of G. No confusion should arise from our abbreviation since no other kind of rank will be used.) Closely allied to the concept of rank is that of independence: A set $\{x_i\}$ of elements of G is *independent* in case $\sum m_i x_i = 0$, $m_i \in I$, implies $m_i = 0$ for all i. In particular, each element in an independent set must have infinite order. A *basis* is a maximal independent subset; all bases have the same cardinality which is equal to the rank.

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