NORMAL ENDOMORPHISMS

BY

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1. Introduction

It should prove helpful to the reader if we begin with some brief remarks not strictly pertaining to the content of the paper. The general topic is the concept and theory of "normal" endomorphisms of a loop. This topic was begun in Chapter IV, §4 of the author's book, *A Survey of Binary Systems* [1]. Most of the references in the paper are to [1], and these take the form "[SIV.4]", "Lemma 3.1 of [SVII]", and so on. The paper is so designed that the reader can follow the earlier sections with only a general knowledge of loop theory but will need progressively more of the lore of Moufang loops. It is, however, the special knowledge which supports the concepts studied in the paper. Accordingly, we shall now discuss the content of the paper with a minimum of definitions and with little regard to order. We adopt a notation appropriate for the discussion, with no intention of using it in the rest of the paper.

The four main classes of "normal" endomorphisms to be studied in this paper are

K_1 : *The seminormal endomorphisms.*

K_2 : *The weakly normal endomorphisms.*

K_3 : *The normal endomorphisms.*

K_4 : *The strongly normal endomorphisms.*

Each of these is defined in §2. Class K_3 was defined in [SIV.4] in terms of "normalized, purely non-abelian" loop words. We use the same definition here but prove some helpful lemmas (Lemmas 2.1, 2.2) about the defining class of loop words. The remaining classes K were studied originally in response to a question (proposed in a letter from Reinhold Baer) as to whether, in the case of a group, K_3 was precisely the set of all endomorphisms commuting with the inner automorphisms. The answer turns out to be affirmative (Corollary to Theorem 3.2), and the introduction of the new classes allows us to place this result among the elementary ones. The method of proof is as follows: For any loop (by definition and Theorem 2.1),

$$K_1 \supset K_2 \supset K_3 \supset K_4. \tag{1.1}$$

For a group, K_1 and K_4 are easily seen to consist of all endomorphisms commuting with every inner automorphism of the group.

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