ON GENERALIZED CONJUGATE CLASSES IN A FINITE GROUP[1]

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Throughout this note G will denote a finite group, and σ a fixed homomorphism of G into itself. σ may be an automorphism of G. Two elements a and b in G will be called *σ-conjugate* if there exists an element x in G such that $a = x^{-1}bx^{\sigma}$. This is an equivalence relation, and all elements in G are partitioned into *σ-classes*. If σ is the identity automorphism, then σ-conjugacy reduces to the "ordinary" conjugacy in groups. A subset S of G will be called *σ-invariant* if $x \in S$ implies $x^{\sigma} \in S$. In this note we shall prove the following:

THEOREM 1. *The number of σ-classes equals the number of σ-invariant classes of conjugate elements in G.*

An interesting feature of the above theorem is that, although the theorem itself does not involve group characters, it does not seem to be proved easily without using group characters. The author has been unable to obtain such a proof.[2] Actually Theorem 1 is an immediate consequence of the following two theorems.

THEOREM 2. *The number of σ-classes in G is equal to the number of σ-invariant irreducible ordinary characters of G.*

THEOREM 3. *Let p be an arbitrary prime number. Then the number of σ-invariant irreducible modular characters (with respect to p) is equal to the number of σ-invariant p-regular classes of conjugate elements in G. In particular, the number of σ-invariant ordinary characters is equal to the number of σ-invariant classes of conjugate elements in G.*

Here, a function $\varphi(x)$ defined on a σ-invariant subset S of G is called *σ-invariant* if $\varphi(x^{\sigma}) = \varphi(x)$ for all $x \in S$; a class of conjugate elements is called *p-regular* if it consists of elements of order prime to p.

Theorem 2 above is a generalization of a result of Ado [1], who proved Theorem 2 for the case where σ is an automorphism. In his proof Ado made use of the inverse mapping σ^{-1}. The method we use here to prove Theorem 2 is a rather trivial modification of Ado's.

Proof of Theorem 2. Let $\chi_1, \cdots, \chi_k$ be all irreducible ordinary characters

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[2] Ruth Rebekka Struik pointed out that such a proof can be obtained easily in case one of the numbers involved in Theorem 1 is 1.