ON THE EXACT NUMBER OF PRIMES LESS THAN A GIVEN LIMIT

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The problem of counting the exact number of primes $\leqq x$, without actually listing them all, dates from Legendre [1] who observed that the number of primes p for which $x^{1/2} < p \leqq x$ is one less than

$$x - \sum_i [x/p_i] + \sum_{i<j} [x/p_i p_j] - \cdots,$$

where $[z]$ denotes, as usual, the greatest integer $\leqq z$, and the p_i range over all the primes less than or equal to $x^{1/2}$. Since then, a large number of writers [2] have suggested variants and improvements of this result. Foremost among these was the astronomer Meissel [3] whose formula (5) is derived below. This formula, and Meissel's obscure derivation of it, is to be found in a number of textbooks in number theory. It is of practical value to our problem because in its "Legendre's sum" the primes extend only as far as $x^{1/3}$. Meissel used his formula to evaluate $\pi(x)$, the number of primes $\leqq x$ for a number of large values of x including

$$\pi(10^7) = 664\,579, \qquad \pi(10^8) = 5\,761\,455, \qquad \pi(10^9) = 50\,847\,478.$$

Others writers with "better" formulas than Legendre's or Meissel's have been content to advocate rather than utilize their results. At any rate, until now no one has made such calculations beyond $x = 10^7$, except N. P. Bertelsen [4] who confirmed Meissel's corrected value of $\pi(10^8)$ and computed $\pi(2 \cdot 10^7)$ and $\pi(9 \cdot 10^7)$. Meissel's calculation of $\pi(10^9)$, made sometime between 1871 and 1885, must be regarded as one of the outstanding single calculations of the 19th century, even though his value is slightly in error. Because of the recent interest in such functions as $\pi(x) - \mathrm{li}(x)$, the writer has been considering the problem of extending Meissel's method so as to reduce the range of the primes in the Legendre sum still further. The fact that we now have high speed computers to do our actual calculations does not relieve us of the responsibility of counting the minutes as Meissel must have counted his weeks. Some preliminary work on the SWAC in 1956 indicated that for very fast machines there is a decided lack of balance in Meissel's formula, most of the time being spent on its Legendre sum.

Notation and general formula

Let m_a denote the product

$$m_a = p_1 p_2 \cdots p_a$$

of the first a primes. We consider the general Legendre sum

$$\phi(x, a) = \sum \mu(\delta)[x/\delta],$$

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