A NEW PROOF OF THE CONVERGENCE THEOREM FOR δ-SUBHARMONIC FUNCTIONS

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Convergence theorems play a fundamental role in the development of the theory of subharmonic functions. The simplest such theorem asserts that the limit of a decreasing sequence of functions subharmonic on a common region (connected open set) is either subharmonic or identically $-\infty$. For an increasing sequence of subharmonic functions the situation is more complicated. It was observed first of all by T. Radó [9, p. 22] that if a sequence of this sort is locally bounded above, then the limit function is almost subharmonic (that is to say, coincides almost everywhere with a function subharmonic on the region). Later, Brelot succeeded in showing [4] that the exceptional set is not only of Lebesgue measure zero but is, in fact, of interior capacity zero. Utilizing the energy norm, Cartan [6] then further refined this result to replace "interior capacity" by "exterior capacity". However, with the advent of the theorem of Choquet [7] to the effect that all Borel sets are capacitable, the latter form of the convergence theorem is immediate from the earlier version given by Brelot.

We shall be concerned here with the extension of the convergence theorem to δ-subharmonic functions, that is, to functions representable as differences of subharmonic functions on a region Ω of k-dimensional Euclidean space. For simplicity, the actual calculations will be carried out only for the case of the classical δ-subharmonic functions on plane regions. It is clear, however, that the same techniques yield corresponding results for wider classes of functions and regions. For example, the classical δ-subharmonic functions can be replaced by functions similarly related to more general potential kernels, and the plane region Ω can be replaced by a region in an arbitrary Green's space.

A weak form of the convergence theorem was first established in [2, pp. 345–346], and we state it here for reference.

THEOREM 1. *Let $\{w_n\}$ be a sequence of functions almost δ-subharmonic on Ω. If the sequence of total variations of the corresponding mass distributions is bounded, and if $\{w_n\}$ converges in the mean locally to a function w on Ω, then*

(i) *w is almost δ-subharmonic on Ω, and*

(ii) *the sequence $\{m_n\}$ of mass distributions for the functions w_n converges weakly*[1] *to the mass distribution m for w.*

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[1] Weak convergence is used here in the sense of Definition 7 of [2, p. 345].