A PROPERTY OF BROWNIAN MOTION PATHS[1]

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1. Introduction

Let $x(t, \omega)$ be a separable one-dimensional Brownian motion process with $x(0, \omega) = 0$. We suppose that, if necessary, a set of sample points of measure zero has been discarded so that all the sample functions of the process are continuous. Then for every t and ω a measure $\mu(\cdot\,, t, \omega)$ is defined by taking $\mu(A, t, \omega)$ to be the Lebesgue measure of the set $\{\tau : 0 \leqq \tau \leqq t, x(\tau, \omega) \in A\}$. In this paper we prove the following result.

THEOREM. *With probability one, $\mu(\cdot\,, t, \omega)$ has a continuous density function. That is, for almost all ω, there exists a function $f(x, t, \omega)$ which is continuous in x and t such that*

$$\mu(A, t, \omega) = \int_A f(x, t, \omega)\, dx$$

for every Borel set A.

The proof occupies Sections 2 and 3. Explicit bounds for the moduli of continuity of f are given by (2.1) and (2.3).

This theorem represents a partial extension of results of P. Lévy connected with the notion of "mesure du voisinage" (of the set of zeros of a Brownian motion sample function) introduced and investigated by him [5, 6]. Let $F(x, \omega) = \mu([-\infty, x], t, \omega)$. Then one of Lévy's theorems may be paraphrased as follows: *For any fixed ξ, $F'(\xi)$ exists with probability one, and is equal to $(\pi/2)^{1/2}$ times the "mesure du voisinage" of the set*

$$A_\xi = \{\tau : \tau \leqq t, x(\tau, \omega) = \xi\}.$$

Our result is stronger on the one hand, in that we show that with probability one, $F'(\xi)$ exists for all ξ simultaneously and is continuous. On the other hand, we do not show any connection between $F'(\xi)$ and the set A_ξ.

We have stated the theorem for the case of Brownian motion, but it can be extended to a very general class of one-dimensional diffusion processes. Let $x(t, \omega)$ be a process such that the infinitesimal generator of the associated semigroup [1] has the form [2]

$$\Omega = \frac{d}{dm}\frac{d}{dx},$$

where m is an arbitrary strictly increasing function. Then $\mu(\cdot\,, t, \omega)$ can be defined as before, and the conclusion is that for almost all ω there is a con-

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