THE STRUCTURE OF UNITARY AND ORTHOGONAL QUATERNION MATRICES

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1. Introduction

It is known that a normal quaternion matrix (and hence a unitary quaternion matrix) is unitarily similar to a diagonal matrix with complex elements [2]. It is also known that a quaternion matrix is unitarily equivalent to a diagonal matrix with nonnegative real elements [4]. Also, the transpose of a unitary quaternion matrix is not necessarily unitary; a necessary and sufficient condition that the transpose, V^T, of a unitary quaternion matrix V be unitary is that there exist real orthogonal matrices U and W such that $UVW = D$ is a diagonal quaternion matrix [5].

In the present work two theorems are obtained concerning the structure of unitary and orthogonal quaternion matrices, respectively. An orthogonal quaternion matrix, P, is defined to be a matrix such that $PP^T = I\ (= P^TP)$, where P^T denotes the transpose of P. In each case the essential "quaternion character" of the matrix is clearly revealed by the form obtained; and in the unitary case the form obtained gives more meaning to the above quoted theorem concerning the transpose of a unitary matrix.

2. The structure of a unitary matrix

The following theorem will be obtained:

THEOREM 1. *Every quaternion unitary matrix P can be written in the form $P = UDW$, where U and W are complex unitary matrices and D is a quaternion diagonal unitary matrix; conversely, every matrix of this form is a quaternion unitary matrix.*

Let $P = P_1 + jP_2$ (where P_1 and P_2 have complex elements) be a unitary quaternion matrix. Then, since $PP^{CT} = I = P^{CT}P$ $\ (P^{CT} = P_1^{CT} - jP_2^T$ denotes the quaternion-conjugate transpose of P), the following hold:

$$P_1P_1^{CT} + P_2^CP_2^T = I = P_1^{CT}P_1 + P_2^{CT}P_2\,,$$

$$P_2P_1^{CT} - P_1^CP_2^T = 0 = P_1^TP_2 - P_2^TP_1\,.$$

By a known theorem [1] for the complex matrix P_1 there exist two complex unitary matrices U_1 and W_1 such that $U_1P_1W_1 = D$ is a real diagonal matrix with nonnegative elements along the diagonal. There is no loss in generality in assuming that like diagonal elements are arranged together so that $D = D_1 \dotplus D_2 \dotplus \cdots \dotplus D_k$ where $D_i = c_iI_i$ where c_i is nonnegative and real,

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