MARKOFF PROCESSES AND POTENTIALS I

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This paper will appear in three parts. Its purpose and scope are best discussed after the state of affairs in the simplest situation has been explained.

Let $P_\tau(r, ds)$ be stationary Markoff transition measures on $\mathcal{K}$, a separable locally compact space, and suppose that the transformations P_τ of functions on $\mathcal{K}$,

$$P_\tau f.(r) \equiv \int P_\tau(r, ds)f(s),$$

leave invariant the Banach space of continuous functions vanishing at infinity and converge strongly there to the identity transformation as $\tau \to 0$. The transition measures can be realized by Markoff processes whose sample paths are continuous on the right and have limits from the left; the letter X will denote such a process, $X(\tau)$ the random point of $\mathcal{K}$ obtained by fixing the time as τ, and $X(\tau, \omega)$ the point of $\mathcal{K}$ obtained by fixing in addition the element ω of the basic probability space over which the process is defined.

A set E in $\mathcal{K}$ is said to be nearly Borel if, for each process X, there are Borel sets A and B such that $A \subset E \subset B$ and such that, for almost all ω and for all τ, the point $X(\tau, \omega)$ belongs to all three sets if it belongs to one. Such sets form a Borel field; a function measurable over the field is said to be nearly Borel measurable. The time T at which a process X hits a nearly Borel set E is defined to be the infimum of the strictly positive τ for which $X(\tau, \omega)$ belongs to E, or ∞ if there are no such τ; it is a random variable, that is to say, a measurable function on the basic probability space. If X starts at a point, T vanishes with probability either 0 or 1. In the latter event the point is said to be regular for E; for example, an interior point of E is certainly regular for E. A point r and a nearly Borel set E determine a measure $H_E(r, ds)$, defined by the formula

$$H_E(r, A) \equiv \mathcal{P}\{X(T(\omega), \omega) \in A, T(\omega) < \infty\},$$

where X is a process starting at r and $\mathcal{P}\{\cdots\}$ stands for the probability of the event within the curly brackets.

A few remarks on language and notation are needed. Let $\mathcal{A}$ be the Borel field comprising the sets in $\mathcal{K}$ that are measurable for the completion of every measure defined on the topological Borel field of $\mathcal{K}$. A function on $\mathcal{K}$ is understood to be measurable over $\mathcal{A}$; a positive function may take on the values 0 and ∞; a measure on $\mathcal{K}$ is understood to be defined on $\mathcal{A}$ and to be the sum of countably many bounded positive measures. A kernel, say

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