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SQUARE INTEGRABLE HARMONIC FORMS AND REPRESENTATION THEORY

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1. Introduction. An interesting class of irreducible unitary representations of semisimple Lie groups consists of the representations associated to elliptic coadjoint orbits. An important open problem is to give a construction of these representations in terms of the geometry of the orbits. For example, one may construct a Hilbert space of sections of a bundle over the orbit, the Hilbert space inner product being a G-invariant L_2 inner product. Consider an elliptic coadjoint orbit $G \cdot \chi = G/L$. As explained in Section 2, G/L has an invariant complex structure and is biholomorphic to an open orbit in a complex flag manifold. Associated to this orbit is the representation of G naturally occurring on the sheaf cohomology space $H^s(G/L, \mathcal{O}(\mathscr{L}_\chi))$ (under some negativity condition on the line bundle $\mathscr{L}_\chi$). It is a (difficult) known result that this gives a continuous representation of G on a Frechet space. In fact, $H^s(G/L, \mathcal{O}(\mathscr{L}_\chi))$ is a maximal globalization in the sense of [18] (see [26]). It follows that the unitary globalization lies inside the cohomology space. Thus, it is reasonable to look for "L_2" representatives of cohomology classes and define an invariant inner product. This space of representatives should give a Hilbert space with an invariant inner product on these representatives. A somewhat simple example occurs when G is a compact group. The orbit then carries a positive-definite hermitian metric (defining notions of L_2 and harmonic), and the Hodge theorem provides a space of L_2-harmonic forms, one from each cohomology class. These L_2-harmonic spaces are realizations of the unitary representations of the compact group.

Because we are interested in arbitrary semisimple Lie groups, often the only invariant hermitian metric available is indefinite. This indefinite metric can be used to define a global invariant hermitian form on $\mathscr{L}_\chi$-valued type-$(0,s)$ differential forms. Of course, the integral defining this global form must converge. In a very general setting, we show in Theorem 3.4 how to choose representatives for each K-finite cohomology class for which the integral defining the global form converges. In the case when G/L is an indefinite Kähler symmetric space (i.e., L is the fixed-point group of an involution), we get the following stronger result. Following the definition in [15], using an auxiliary metric on G/L that is positive definite but not G-invariant, we define a Hilbert space of square-integrable $\mathscr{L}_\chi$-valued type-$(0,s)$ differential forms representing for all K-finite cohomology classes. On this L_2-space the integral defining the global invariant form is con-

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