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A SHORT PROOF OF THE INTEGRALITY OF THE MACDONALD (q,t)-KOSTKA COEFFICIENTS

LUC LAPOINTE AND LUC VINET

1. Introduction and background. Let Λ_N denote the ring of symmetric functions in the variables $x_1, x_2, \ldots, x_N$, and denote by $\mathbb{Q}(q,t)$ the field of rational functions of the parameters q and t with rational coefficients. The Macdonald polynomials (see [13]) $J_\lambda(x;q,t)$ are symmetric polynomials labelled by partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$, that is, sequences of nonnegative integers in decreasing order. These polynomials form a basis for $\Lambda_N \otimes \mathbb{Q}(q,t)$ and can be characterized as the joint eigenfunctions of the commuting operators $\{M_N^k, k = 0, \ldots, N\}$ defined as follows:

$$M_N^k = \sum_I t^{(N-k)k+k(k-1)/2} \tilde{A}_I(x;t) \prod_{i \in I} T_{q,x_i} \tag{1}$$

with $M_N^0 = 1$. Here, the sum goes over all k-element subsets I of $\{1, \ldots, N\}$

$$\tilde{A}_I(x;t) = \prod_{\substack{i \in I \\ j \in \{1,\ldots,N\} \setminus I}} \frac{x_i - t^{-1} x_j}{x_i - x_j}, \tag{2}$$

and T_{q,x_i} stands for the q-shifted operator in the variable x_i ($T_{q,x_i} f(x_1, \ldots, x_i, \ldots) = f(x_1, \ldots, qx_i, \ldots)$). This notation will be used throughout the paper.

The eigenvalue equations that the Macdonald polynomials satisfy are conveniently written in terms of the generating function

$$M_N(X;q,t) = \sum_{k=0}^{N} M_N^k X^k, \tag{3}$$

where X is an arbitrary parameter. With J, a set of cardinality $|J| = j$, we shall also use the notation $M_J(X;q,t)$ to designate $M_j(X;q,t)$ in the variables x_i, $i \in J$. If $\ell(\lambda)$ denotes the number of nonzero parts of λ, for $\ell(\lambda) \leqslant N$ we have

$$M_N(X;q,t) J_\lambda(x;q,t) = a_\lambda(X;q,t) J_\lambda(x;q,t) \tag{4}$$

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