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# $L^2$ ESTIMATES FOR AVERAGING OPERATORS ALONG CURVES WITH TWO-SIDED $k$-FOLD SINGULARITIES

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**§0. Introduction.** In this note we prove in high dimensions a result about averaging operators along curves first proved for $\mathbb{R}^2$ by Phong and Stein [7], [8], using their same methods.

The issues raised here are part of the general problem of finding sharp Sobolev estimates for Fourier integral operators (FIOs) whose canonical relation is not a canonical graph.

We recall that to any FIO from, say, $C_0^\infty(X) \to \mathscr{D}'(Y)$, for $X$ and $Y$ two given manifolds, is associated a Lagrangian submanifold of $T^*(X \times Y)$—let us call it $\mathscr{L}$—which contains the interesting part of the information about the regularity properties of the operator (see [2]). We say that an FIO is nondegenerate when the left and right projections $d\pi_L$ (resp., $d\pi_R$) of $\mathscr{L}$ on $T^*X$ (resp., $T^*Y$) are local diffeomorphisms. $L^2$ estimates for nondegenerate FIOs can be found in [2], $L^p$ estimates in [14]. Even though there is a general result about $L^2$ estimates for degenerate FIOs (see [2] and [3, p. 30]), much remains to be understood about sharp estimates for special classes of degenerate FIOs. The best-known operators of this kind are those where the maps $d\pi_L$ and $d\pi_R$ have folds (or $k$-folds with $k = 1$); that is, the (common) singular set is a submanifold $\Sigma$ of codimension 1 in $\mathscr{L}$ transversal to the kernel of the derivatives of $d\pi_L$ and $d\pi_R$, and the determinants are zero to the order $k$ for $k = 1$ in $\Sigma$. (If $k \geqslant 1$, we have the so-called $k$-folds (see [8]).) For such operators, $k = 1$, $L^2$ estimates can be found in [4] and $L^p$ estimates in [7], [8], [13], and [15]. For FIOs $C_0^\infty(\mathbb{R}^2) \to \mathscr{D}'(\mathbb{R}^2)$ with averaging operators, the Lagrangian $\mathscr{L}$ is a conormal bundle. Other more degenerate cases are discussed in the series by Phong and Stein [8]–[12] and by Seeger [13], among them those with $k$-folds with $k > 1$.

For the case when only one of the two projections $d\pi_L$, $d\pi_R$ has singularities of some special type and there are no hypotheses on the singularities of the other, we refer, for instance, to [1], [8], [13], and the references therein.

Here we prove sharp $L^2$ estimates for FIOs $C_0^\infty(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ with $k$-folds and where the Lagrangian is the conormal bundle of a submanifold of codimension $n - 1$ in $\mathbb{R}^n \times \mathbb{R}^n$. We are able also to obtain a new proof of a classical result of Melrose and Taylor [4]—that is, the case $k = 1$—without the assumption that the Lagrangian is a conormal bundle.

As we already mentioned, what we do here arises naturally from work by

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