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PERIODICITY OF THE FIXED LOCUS OF MULTIPLES OF A DIVISOR ON A SURFACE

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In his paper "The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface" [Z], Zariski studies the Riemann-Roch problem on an algebraic surface. His main results are presented in the final section of his paper, titled "A Summary of Principal Results".

Let D be an effective divisor on a nonsingular projective surface S over an algebraically closed field k. Zariski shows that $\dim_k |nD|$ is a quadratic polynomial in n plus a bounded function of n. Zariski shows that this bounded function is eventually periodic in some cases, and he asks if the function is eventually periodic for an arbitrary effective divisor on S.

This question is answered in [CS]. We give an affirmative answer to this question when k has characteristic zero [CS, Thm. 2], and we provide a counterexample in positive characteristic [CS, Ex. 3].

Zariski also considers the behaviour of the fixed locus of $|nD|$ for large n. The fixed component of D is the largest effective divisor E such that $L - E$ is effective for all L in the complete linear system $|nD|$. Let B_n be the fixed component of $|nD|$. Zariski shows that there exists a divisor E on S such that $B_n - nE$ is bounded. Zariski proves that $B_n - nE$ is eventually periodic in some cases. It is natural to ask if this divisor is eventually periodic for an arbitrary effective divisor on S.

In this paper (Theorem 5) we give a positive answer to this question. We prove that the divisor $B_n - nE$ is periodic in n for an arbitrary effective divisor D on a surface S if k has characteristic zero. However, [CS, Ex. 3] shows that periodicity of the fixed component does not hold in positive characteristic.

An example is given in Section 3 of this paper showing that the codimension-two part of the base locus can be nonperiodic, even on a surface in characteristic zero.

An example showing nonpolynomial-like growth of the fixed locus for divisors on a three-fold is given in [CS, §7].

Our main technical tool in proving Theorem 5 is [CS, Thm. 8], a periodicity theorem on cohomology. This is stated in Theorem 4 in Section 1 of this paper.

We will use the following notation. A Cartier divisor D (or a line bundle $\mathscr{L}$) on a nonsingular projective variety X is nef if D (respectively $\mathscr{L}$) has a nonnegative intersection number with every curve on X. For any proper k-scheme X, a line bundle $\mathscr{L}$ on X is numerically trivial if its restriction to every integral curve in X has degree 0. A $\mathbf{Q}$-divisor is a rational sum of integral (prime) divisors.

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