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NWO SEQUENCES, WEIGHTED POTENTIAL OPERATORS, AND SCHRÖDINGER EIGENVALUES

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1. Introduction and summary. In earlier papers Stephen Semmes and the author used nearly weakly orthonormal (NWO) sequences as a technical tool for obtaining boundedness criteria and eigenvalue estimates for classes of singular integral operators ([R1], [RS1, 2, 3]). Here that approach is used to study weighted norm inequalities for potential operators and the closely related problem of eigenvalue estimates for Schrödinger operators. We find that for weights and potentials which have some smoothness (Muckenhoupt conditions) NWO sequences can be used to give very straightforward proofs of a number of results.

In Section 2 we collect background information about NWO sequences and indicate how the techniques of [RS2] extend to operators mapping L^p to L^q, $p < q$. (In fact, in that case the techniques simplify slightly.)

In Section 3 we prove weighted norm inequalities for the Riesz potentials and the Bessel potentials and also obtain weighted Sobolev and Poincaré inequalities.

In Section 4 we look at eigenvalue estimates for Schrödinger operators. Our starting point is the observation by Birman and Schwinger that the number of eigenvalues of a Schrödinger operator below a cutoff can be estimated in terms of the number of large eigenvalues of an associated weighted potential operator. An analysis similar to that in Section 3 is then given for the potential operator, and we read off the desired estimates. Our conclusions are estimates in the style of C. Fefferman and Phong [F] that the number of eigenvalues in a certain range can be estimated in terms of the number of disjoint dyadic cubes on which weighted means of the potential are large.

The analysis in Section 4 is based on an explicit decomposition of a compact weighted potential operator as a finite-rank piece plus a small remainder. This allows explicit inversion of the Lipmann-Schwinger equation for the generalized eigenfunctions of the Schrödinger operator. We also have a series representation, similar to the Born series, for the actual eigenfunctions. This is discussed in Section 5.

The techniques we are using make it quite easy to draw conclusions about membership of operators in Schatten-von Neumann ideals. We give an example of that in Section 6 by developing such criteria for commutators of multiplication and potential operators acting on weighted L^2 spaces. We then examine the same question for the operator which occurs in the Lipmann-Schwinger equation. Because that operator has an oscillatory kernel, the techniques of [RS2] do not apply

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