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ON THE IMAGE SIZE OF SINGULAR MAPS II

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Introduction. The basis of numerous investigations of the geometry of critical sets and critical values has been the construction by H. Whitney [W] of a C^1 real-valued function nonconstant on a connected set of critical points. The central problem posed by Whitney's example is to determine sufficient conditions under which the images of critical sets have Lebesgue measure zero or, more generally, Hausdorff measure zero of an appropriate dimension. Important early work of M. Morse, A. P. Morse [M], and A. Sard [Sa1] identified the differentiability class of a mapping as one such criterion: By the classical Morse-Sard theorem (see [Stb]), the image of *any* critical set of a map $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$ has Lebesgue m-measure zero provided $k \geqslant max\{n - m + 1, 1\}$.

Subsequent approaches to the problem of critical image size have improved the Morse-Sard smoothness hypothesis for specific classes of critical sets. On the one hand, Sard [Sa4], Federer [Fed], and Yomdin [Y1] have derived a sharp upper bound for the Hausdorff and entropy dimension of the image of rank-r sets (see definitions below) in terms of the differentiability class of f. A dual approach, initiated by Sard [Sa3] and recently completed by Norton [Nor1, Nor2, Nor3], relates the Hausdorff dimension of a rank-r set to a precise Hölder smoothness condition on f which insures that its image under f has Lebesgue measure zero.

A common feature of the preceding results is the absence of any restrictive assumptions regarding the nondegeneracy of the derivative Df *near* the critical set in question. A natural question is thus to what extent critical image size depends on the presence of regular points. In Whitney's construction, for example, regular points clearly play an essential role, since any real-valued function is constant on a connected open set of critical points. A stark contrast to this situation, however, is provided by examples due to Kaufman [Ka], Yomdin [Y1], and the author [Ba1] of highly differentiable vector-valued mappings which are everywhere critical and yet whose images have nonempty interior!

The purpose of this note is to answer the preceding question by introducing several image-size theorems for singular (i.e., everywhere critical) maps. Our discussion is divided into two parts. In §1 we define the *subrank* of a singular map $f: \mathbb{R}^n \to \mathbb{R}^m$ and use this quantity to derive a general smoothness condition which implies $f(\mathbb{R}^n)$ has m-measure zero. This leads in particular to a natural analog of the Morse-Sard theorem for the class of singular maps in which we significantly weaken the differentiability requirement of the original theorem. Subsequently, we

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