Vol. 68, No. 1 DUKE MATHEMATICAL JOURNAL © October 1992

CONTINUATION TO THE MINIMAL NUMBER OF CRITICAL POINTS IN GRADIENT FLOWS

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1. Introduction. A central idea in the theory of the Conley index is the notion of continuation. Roughly, given a one-parameter family of flows and a compact set N which is an isolating neighborhood for each flow in the family, then the sets isolated by N at any two-parameter values are said to be related by continuation. An obvious question to ask is when two isolated invariant sets are related by continuation. The Conley index can be used to give a negative answer to this question: two sets which are related by continuation have the same Conley index. Conley asked whether the converse is true, i.e., are any two sets with the same index related by continuation? We cannot give a complete answer to this question, but we can outline a procedure which gives a partial answer to the question. The procedure uses the "cancellation" of critical points in gradient-like flows.

Suppose S is an isolated invariant set in a flow which is gradient-like and Morse-Smale (i.e., the critical points of the flow are nondegenerate, and stable and unstable manifolds intersect transversally). Then S consists of a finite number of critical points, plus connecting orbits between the critical points. One can make a free chain complex (C_k, ∂_k) as follows: C_k has one generator for each critical point, and ∂_k from a generator of index k to one of index $k-1$ counts the number of connecting orbits between the critical points (with orientation). A fundamental result going back to M. Morse is that $H(C_k, \partial_k)$ is isomorphic to the homology Conley index of S. This gives a lower bound on the number of critical points in any continuation. The main idea in this paper is the use of the converse of this fact, namely that, if certain topological assumptions are satisfied, then given any chain complex (C_k, ∂_k) with $H(C_k, \partial_k)$ isomorphic to the homology index of S, the set S can be continued to a set where the number of critical points of index k equals the number of generators of C_k. In the case where S is a manifold and the homology index is 0, this is essentially the h-cobordism theorem due to Wallace ([Wa]) and Smale ([Sm2]). We modify the proof in Milnor's book [Mi], and the notes of Franks [F] to prove the theorem in our more general setting.

If S is an isolated invariant set in a flow which is not gradient-like, then we can continue it to a set in a flow that is, and so the above discussion applies to general isolated invariant sets; i.e., we can continue to any collection of critical points provided they generate a chain complex with the correct homology. This fact has some interesting corollaries; e.g., any set of homology index 0 can be continued to

Received 9 May 1991. Revision received 23 May 1992.
Research supported in part by the National Science Foundation.