Vol. 63, No. 1 DUKE MATHEMATICAL JOURNAL © June 1991

ON ISOSPECTRAL POTENTIALS ON TORI

CAROLYN S. GORDON AND THOMAS KAPPELER

1. Introduction. Given a square integrable function q on a flat torus $T = \mathbb{R}^d/\mathscr{L}$, denote by spec($q$) the eigenvalue spectrum of the Schrödinger operator $-\Delta + q$ and set $\mathrm{Iso}(q) = \{q' \in L^2(T)\colon \mathrm{spec}(q) = \mathrm{spec}(q')\}$, the isospectral set of q. In case $d = 1$ (i.e., T is a circle), all the isospectral sets are well known; generically, they are infinite-dimensional tori. In dimension $d \geqslant 2$ there is evidence that the isospectral sets are small. However, the only isospectral sets known are those of the constant potentials. They are uniquely determined by their spectra. A natural starting point in studying dimensions $d \geqslant 2$ is to consider Iso(q) for q a completely separable potential on a rectangular torus $T = \mathbb{R}^d/(a_1\mathbb{Z} \times \cdots \times a_d\mathbb{Z})$. We say q is completely separable if q can be written in the form $q = \sum_{i=1}^d q_i(x_i)$ where $q_i \in L^2(\mathbb{R}/a_i\mathbb{Z})$. For such potentials q, $\sum_{i=1}^d \mathrm{Iso}(q_i) \subseteq \mathrm{Iso}(q)$ where $\mathrm{Iso}(q_i)$ is the isospectral set of the potential q_i in $L^2(\mathbb{R}/a_i\mathbb{Z})$. Thus in this case Iso($q$) being small means that the above inclusion is in fact an equality. Therefore, we ask

(Q) does $\mathrm{Iso}(q) = \sum_{i=1}^d \mathrm{Iso}(q_i)$?

Question (Q) is equivalent to the following two questions.

(Q1) If q is completely separable and $\mathrm{spec}(\tilde{q}) = \mathrm{spec}(q)$, is $\tilde{q}$ completely separable?

(Q2) If q and $\tilde{q}$ are isospectral, completely separable potentials, are the one-dimensional potentials q_i and $\tilde{q}_i$ isospectral, $1 \leqslant i \leqslant d$ (up to possible permutations)?

Our interest in this problem is motivated by a paper of Eskin, Ralston, and Trubowitz [ERT] which addresses question (Q1). They answer positively question (Q1) for generic rectangular tori by constructing certain spectral invariants involving a decomposition of q into a sum of one-dimensional potentials. (See also [MN] and [S].)

One of our main results is an affirmative answer to (Q) in dimension $d = 2, 3$ for a large class of rectangular tori, including all rational tori. The extra hypothesis on the tori is used only for question (Q2) and can be dropped completely in case $d = 2$ if we impose a mild regularity condition on the potentials.

Received 15 May 1989. Received revision 21 May 1990.
Gordon partially supported by NSF grant # 8502084. Kappeler partially supported by NSF grant.