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HEIGHTS IN FAMILIES OF ABELIAN VARIETIES

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1. Introduction. Suppose we are given K a number field, X a projective scheme over K, Y an open subscheme of X, A/Y an abelian scheme with section s, $\mathscr{L}$ an invertible sheaf on A. Consider Y as a scheme over $\mathbb{Q}$. Every $\bar{\mathbb{Q}}$-valued point x of Y is defined over some number field L and we get by base extension an abelian variety A_x over L, a section s_x, and an invertible sheaf $\mathscr{L}_x$.

Let $\hat{h}(s_x)$ denote the Néron-Tate height of s_x in A_x with respect to $\mathscr{L}_x$. Then $\hat{h}(s_x)$ is independent of the choice of L, so it defines a real-valued function h_A on $Y(\bar{\mathbb{Q}})$. h_A is the sum of two functions, h_A^q and h_A^l, corresponding to the quadratic and linear parts of the Néron-Tate height. If X is normal, then associated with any invertible sheaf $\mathscr{M}$ on X there is the Weil height $h_{\mathscr{M}}$ on $X(\bar{\mathbb{Q}})$, defined only up to a bounded function.

For definitions and properties of these heights see [La]. In this paper I establish an expression for h_A^q in terms of such a height when X is a normal connected curve. The precise statement will be given below.

The result was initially conjectured by Silverman for the case of an elliptic family over the projective line. He also derived an asymtotic expression in [Si]. The result for general elliptic families was proved by Tate in [Ta]. Lang extended Tate's proof to abelian families after assuming the existence of "good" completions of Néron models (see chapter 12 of [La]). Under the same assumption Call proves in [Ca] a corresponding result for the local height functions, as well as several weaker results without this assumption. The proof given in this paper is an application of the compactification of the moduli stack of abelian varieties constructed by Faltings in [Fa1] and in [Fa2]. In [Ch] Chai subsequently constructed a compactification of the universal abelian scheme, but this will not be needed here. It remains to be considered whether the techniques used in this paper would give the results of Call on local heights in the general case.

From now on X will be a normal connected curve which is proper and consequently projective over K. This implies that Néron models exist over X. For this and other properties of Néron models see [Ar]. Let N be a Néron model of the generic fiber of A/Y and let N° be its connected component. Note that N is of finite type and smooth over X and N° is an open subscheme of N. Both are noetherian, regular, and integral. Over Y, A is a Néron model and therefore isomorphic to the restrictions of both N and N°. Identify A with these restrictions. From the definition of Néron model, s can be uniquely extended to a section of N/X. Let s also denote this section. Except over a finite subset in the base, it is already a section of N°/X.

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