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AN ISOPERIMETRIC INEQUALITY WITH APPLICATIONS TO CURVE SHORTENING

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1. Introduction. In this note we prove the isoperimetric inequality

$$\pi \frac{L}{A} \leqslant \int_0^L \kappa^2 \, ds$$

for closed, convex C^2 curves in the plane. L, A and κ are the length of the curve, the area it encloses, and its curvature. The inequality does not necessarily hold for nonconvex curves. We use the inequality to show that when a convex curve is deformed along its normal at a rate proportional to its curvature the isoperimetric ratio L^2/A decreases. In some sense the curve is becoming more circular.

Notation and useful formulae are described in the next section, the isoperimetric inequality is proved in section 3 and the application to curve shortening is sketched in section 4.

I wish to thank E. Calabi, for bringing the isoperimetric problem to my attention and pointing out its relevance to curve shortening, and C. Croke who observed that the problem could be reduced to proving inequality A below. I also wish to thank H. Gluck for several useful conversations on the curve shortening problem and the referee for his suggestions on the presentation of these results.

2. Notation. We let $X(s)$ describe the closed, convex curve γ, with arclength parameter s. $T(s)$ and $N(s)$ represent the unit tangent and inward normal vectors which form a frame whose orientation agrees with that of the plane. The curvature at s is denoted $\kappa(s)$ and the support function $p(s) = \langle X, -N \rangle$. L and A are the length of γ and the area of the lamina it encloses.

The length and area can be expressed in terms of the support function and the curvature: From Green's theorem one derives

$$A = \frac{1}{2} \int_0^L (xy' - yx') \, ds = \frac{1}{2} \int_0^L \langle X, -N \rangle \, ds = \frac{1}{2} \int_0^L p \, ds. \tag{1}$$

Using $X'(s) = T(s)$ and $T'(x) = \kappa N(s)$ we obtain

$$\begin{aligned} \int p\kappa \, ds &= -\int_0^L \langle X, \kappa N \rangle \, ds = -\int_0^L \langle X, X'' \rangle \, ds \\ &= -\langle X, X' \rangle \Big|_0^L + \int_0^L \langle X', X' \rangle \, ds \\ &= L. \end{aligned} \tag{2}$$

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