TRANSLATIONAL REPRESENTATIONS OF ONE-PARAMETER SEMIGROUPS

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1. Introduction. The purpose of this note is to prove a rather general variant of Cooper's well-known theorem on isometric semigroups in Hilbert space. Recall that this theorem says that any strongly continuous semigroup of isometries on a Hilbert space may be written as a direct sum of a semigroup of unitaries and the semigroup of right translations acting on $L^2(K)$ where K is some auxiliary Hilbert space (see [1], [3], [6], [8], [9]).

In the situation we envision, the Hilbert space is replaced by an arbitrary Banach space and the condition that the semigroup be isometric is replaced by the hypothesis that the semigroup possess an attendant semigroup of left-inverses (in the classical case, the adjoint semigroup). This situation is also discussed somewhat in the book of Gohberg and Feldman for the special case where the semigroup consists of what they call strict-isometries (see [4], pp. 47–52). In addition we need a hypothesis to ensure that the semigroup consists purely of translations. In the classical case, this is accomplished by demanding that the kernels of the adjoint semigroup fill out a dense manifold and this is the condition we will impose. It turns out that these two conditions alone suffice to afford us a translational representation.

The proof that we give is extremely elementary and depends only on the most rudimentary facts about semigroups, such as can be easily obtained from ([2] pp. 613–627). In particular, detours into the spectral domain are avoided as is the use of Cayley transform arguments.

For Hilbert space, there is a well-known characterization of one-parameter translation groups due to Sinai which is readily deduced from Cooper's Theorem (see [9] p. 152). Sinai's Theorem plays an important role in the Lax–Phillips Theory of scattering [5], and is readily shown to be equivalent to von Neumann's unicity Theorem for solutions of the Weyl commutation relations [7]. We will show that Sinai's Theorem has a Banach-space analogue which is easily recovered from our representation theorem for semigroups.

Finally, we formulate and prove an analogue of a theorem of the author concerning semigroups of partial isometries. Even in the Hilbert space case, the complete description of such semigroups is rather complicated (see [10] and [11]) and the notions needed to describe them do not seem to have Banach space analogues, so we content ourselves with the simplest case.

If X is a Banach space, $\mathcal{L}(X)$ will denote the algebra of bounded operators on X. H will be reserved for complex Hilbert spaces. Subspaces are auto-

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