ON THE EXTENSION OF TURAN'S INEQUALITY TO JACOBI POLYNOMIALS

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1. Introduction. During the 1940's, while investigating the zeros of Legendre polynomials $P_n(x)$, P. Turán [10] observed that

$$P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0, \qquad -1 \leq x \leq 1, \tag{1.1}$$

with equality only for $x = \pm 1$. Shortly thereafter many proofs of (1.1) appeared, and analogous results were obtained for ultraspherical, Laguerre, and Hermite polynomials and for Bessel functions [3; 209]. In 1962 Szegö [8] extended (1.1) to a large class of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. He showed that if $\beta \geq |\alpha|$, $\alpha > -1$, and $R_n(x; \alpha, \beta) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$, then

$$\Delta_n(x; \alpha, \beta) = R_{n+1}^2(x; \alpha, \beta) - R_n(x; \alpha, \beta)R_{n+2}(x; \alpha, \beta) > 0, \qquad -1 < x < 1. \tag{1.2}$$

In addition he conjectured that (1.2) also holds for the triangle

$$U = \{(\alpha, \beta): -1 < \alpha < 0, \alpha < \beta < -\alpha\}.$$

The main purpose of this paper is to prove the following theorem in which we confirm Szegö's conjecture for most of the triangle U.

Theorem 1. *Let $\Delta_n(x; \alpha, \beta)$, $n = 0, 1, \cdots$, be defined as in (1.2) and let*

$$V = \{(\alpha, \beta): \beta \geq \alpha > -1, (\beta - \alpha)(\alpha + \beta)(4\beta^2 + 4\alpha + 1) \geq 0\}.$$

Then

$$\Delta_n(x; \alpha, \beta) > 0 \text{ when } -1 < x < 1 \text{ and } (\alpha, \beta) \in V, \tag{1.3}$$

$$\Delta_n(1; \alpha, \beta) = 0 \text{ when } \alpha, \beta > -1, \tag{1.4}$$

$$\Delta_n(-1; \alpha, \beta) \begin{cases} >0, & \beta > \alpha > -1, \\ =0, & \beta = \alpha > -1, \\ <0, & -1 < \beta < \alpha. \end{cases} \tag{1.5}$$

Our set V contains the line $\alpha = \beta$ (ultraspherical case), the set $\{(\alpha, \beta): \beta \geq |\alpha|, \alpha > -1\}$ considered by Szegö and that part of U which is on or to the left of the parabola $4\beta^2 + 4\alpha + 1 = 0$; see the figure.

From Theorem 1 we derive

Corollary 1. *Let $p_n(x; \lambda) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$, where $P_n^{(\lambda)}(x)$ denotes the*

Received June 23, 1969. Research supported by the National Research Council of Canada.