ON RINGS OVER WHICH SYMMETRIC MATRICES ARE DIAGONALIZABLE

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1. Introduction. If R is a commutative ring with unit element and R_n is the ring of n-by-n matrices with entries in R, call an element A of R_n *symmetric* if $A = A'$, where A' is the usual transpose of A. A fundamental property of the field of real numbers is that every real symmetric matrix is not only congruent, but actually orthogonally congruent and hence similar, to a real diagonal matrix. It is the purpose of this paper to investigate the structure of commutative noetherian rings having this latter matrix property without the strongly geometric requirement of orthogonality.

Definition. If n is an integer ≥ 2, a commutative ring R with unit element is a *ds(n)-ring* if every symmetric matrix in R_n is similar in R_n to a matrix in diagonal form. R is a *ds-ring* if it is a $ds(n)$-ring for every $n = 2, 3, \cdots$.

We shall show first that a ring which is a direct sum of homomorphic images of single-variable formal power series rings over real closed fields is a ds-ring. Then, although no complete characterization is obtained, we shall demonstrate that a commutative neotherian ring which is a $ds(n)$-ring for some particular n must have a structure closely related to this and can in particular be imbedded naturally in such a ring. The relationship of ds-rings to those rings, already characterized by the author [5], over which every matrix is equivalent to a diagonal matrix is also discussed.

2. A class of Ds**-rings.** We first record a useful observation, suppressing most of its straightforward proof.

Lemma 1. *Direct sums and summands of $ds(n)$-rings are $ds(n)$-rings. If R is a $ds(n)$-ring and I is an ideal of R then the residue ring R/I is a $ds(n)$-ring. If R is a $ds(n)$-ring and P is a prime ideal of R then the localization R_P is a $ds(n)$-ring. In particular, if an integral domain is a $ds(n)$-ring, so is its field of quotients.*

Proof. We prove only the statement about localizations. Let R be a $ds(n)$-ring, and P a prime ideal of R. Let $A \in (R_P)_n$ be symmetric. Then $A = (a_{ij}/e_{ij})$, where we may assume that $a_{ij} = a_{ji}$ and $e_{ij} = e_{ji}$ for all i and j. Since any finite set of elements in a localization R_P may be written with a common denominator, we may assume A to have the form (b_{ij}/e), where (b_{ij}) is a symmetric matrix in R_n. Since R is a $ds(n)$-ring, there is a diagonal matrix (d_{ij}) and an invertible matrix (c_{ij}) so that $(c_{ij})(b_{ij}) = (d_{ij})(c_{ij})$ in R_n. This equality leads

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