CHARACTERIZING THE TOPOLOGY BY THE CLASS OF HOMEOMORPHISMS

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In 1958, Everett and Ulam [1] [7] proposed the following problem. Given the class of all homeomorphisms $H(X, \mathfrak{U})$ of a topological space $(X, \mathfrak{U})$ onto itself, what other topologies $\mathfrak{V}$ exist on X such that $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$? We have constructed in [3], [4] and [5] many topologies $\mathfrak{V}$ for X with the same class of homeomorphisms as $(X, \mathfrak{U})$. Now we study this problem from a different point of view. Given a space $(X, \mathfrak{U})$ with the class of homeomorphisms $H(X, \mathfrak{U})$, let $(X, \mathfrak{V})$ be a topological space such that $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$ and which satisfies some additional conditions. Then what can we say about the topology $\mathfrak{V}$?

Whittaker [8] studied a related problem and proved the following theorem.

THEOREM 1. *Suppose $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ are compact, locally Euclidean manifolds, with or without boundary, and suppose α is a group isomorphism between $H(X, \mathfrak{U})$ and $H(Y, \mathfrak{V})$. Then there exists a homeomorphism β of $(X, \mathfrak{U})$ onto $(Y, \mathfrak{V})$ such that $\alpha(h) = \beta h \beta^{-1}$ for all h in $H(X, \mathfrak{U})$.*

Thus we have the following corollary.

COROLLARY 2. *Suppose $(X, \mathfrak{U})$ and $(X, \mathfrak{V})$ are compact, locally Euclidean manifolds, with or without boundary. If $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$, then $(X, \mathfrak{U})$ is homeomorphic to $(X, \mathfrak{V})$.*

We are going to weaken the condition that $(X, \mathfrak{V})$ is a compact, locally Euclidean manifold and require that $H(X, \mathfrak{U}) = H(X, \mathfrak{V})$. The Lemma 3 is known, and its proof can be found in [8] and [2]; the proof of Lemma 4 will appear in [6].

LEMMA 3. *Let U be an open unit ball in an n-dimensional Euclidean space $(E_n, \mathfrak{U})$ and let p, q be points of U. Then there exists a homeomorphism f of $(E_n, \mathfrak{U})$ onto itself such that f is the identity outside U and $f(p) = q$.*

LEMMA 4. *Let A be an open subset of the open unit ball U in $(E_n, \mathfrak{U})$ with center p_0 such that $p_0 \in$ Bndry (A) and $n \geq 2$. Then there exists a finite family of homeomorphisms $g_1, g_2, \cdots, g_m$ of $(E_n, \mathfrak{U})$ onto itself such that*

$$\{p: 0 < d(p_0, p) < \gamma\} \subset \cup \{g_i(A): i = 1, 2, \cdots, m\}$$

for some $\gamma < 1$ and g_i is fixed at p_0 and outside U for each i.

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