LOCALLY COMPACT EQUICHARACTERISTIC SEMILOCAL RINGS

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What conditions on a locally compact ring imply that it is the topological direct product of finitely many algebras over locally compact fields? In investigating this question we shall confine ourselves to equicharacteristic semilocal rings. In §1, a complete answer is given for those rings that have no proper open ideals; in §2 an answer is given for certain noetherian rings that have proper open ideals.

1. Locally compact equicharacteristic semilocal rings. Here we shall prove that a locally compact equicharacteristic semilocal ring none of whose maximal ideals is open is the topological direct product of finitely many finite-dimensional local algebras over indiscrete locally compact fields.

Definition. A *semilocal* ring is a commutative ring with identity that possesses only finitely many maximal ideals. A *local* ring is a commutative ring with identity that possesses only one maximal ideal. An *equicharacteristic* ring is a commutative ring with identity such that for every maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ has the same characteristic as A.

We do not require that a semilocal ring be noetherian nor that the intersection of the powers of its radical be the zero ideal. An algebra over a field that, regarded simply as a ring, is a semilocal [local] ring is, of course, called a semilocal [local] algebra.

Lemma 1. *If $\mathfrak{a}$ is a proper closed ideal of a locally compact equicharacteristic semilocal ring A, then $A/\mathfrak{a}$ is a locally compact equicharacteristic semilocal ring that has the same characteristic as A.*

Proof. As the canonical epimorphism from A onto $A/\mathfrak{a}$ is continuous and open, $A/\mathfrak{a}$ is locally compact. The maximal ideals of $A/\mathfrak{a}$ are the ideals $\mathfrak{m}/\mathfrak{a}$ where $\mathfrak{m}$ is a maximal ideal of A containing $\mathfrak{a}$; consequently $A/\mathfrak{a}$ is semilocal, and as $(A/\mathfrak{a})/(\mathfrak{m}/\mathfrak{a})$ is isomorphic to $A/\mathfrak{m}$ for any ideal $\mathfrak{m}$ containing $\mathfrak{a}$, $A/\mathfrak{a}$ is also equicharacteristic and has the same characteristic as A.

Lemma 2. *A topological semilocal ring* [*algebra*] *whose radical is nilpotent is the topological direct product of finitely many local rings* [*algebras*] *each having a nilpotent maximal ideal.*

Proof. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the distinct maximal ideals of a topological semilocal ring A, and let $\mathfrak{r} = \bigcap_{k=1}^{n} \mathfrak{m}_k$ be the radical of A. By the Chinese Remainder Theorem [13, Theorem 31; 177], there exist $u_1, \ldots, u_n \in A$ such

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