A DENSITY THEOREM FOR OPERATOR ALGEBRAS

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1. Introduction. This paper concerns transitive algebras of operators on Hilbert space, that is, algebras that have no closed invariant subspaces other than the two obvious ones. If the space is finite dimensional, the situation is completely described by a theorem of Burnside [6; 276]: the only transitive algebra is the full algebra of all linear transformations. In the infinite dimensional case, the double commutant theorem shows that every self-adjoint transitive algebra is strongly dense in the algebra of all bounded linear operators. Little is known, however, about non self-adjoint transitive algebras; for example, the invariant subspace problem asks if even a singly generated algebra acting on a separable space $\mathcal{H}$ can be transitive.

While there are no examples and only a few bits of evidence, ones intuition expects that a "topological" Burnside's theorem is probably not true in separable spaces; that transitive operator algebras need not be strongly dense in the algebra $B(\mathcal{H})$ of all bounded operators. It is of interest, then, to know how one might strengthen the hypothesis so as to get a provable theorem. The main result of this paper is that if $\mathcal{A}$ is a transitive subalgebra of $B(\mathcal{H})$ which contains a self-adjoint maximal abelian von Neumann algebra, then $\mathcal{A}$ is strongly dense in $B(\mathcal{H})$ (Theorem 3.3). Then we go on to discuss an example of a different kind; using special features of this example, we obtain a similar theorem.

In §2, we obtain some conditions that a transitive non-dense algebra would have to satisfy (Corollary 2.5). These results are quite general. Theorems 3.3 and 4.4 are then proved by showing that the algebras in question do not satisfy the conditions.

Regarding terminology, P. R. Halmos has pointed out that there is some ambiguity in the literature regarding the term "irreducible" when applied to non self-adjoint operator algebras on Hilbert space: sometimes it means no nontrivial closed invariant subspaces and sometimes it means no commuting self-adjoint projections other than 0 and I. So we have introduced the term *transitive* for the former (a contraction of the term *1-fold transitive* from linear algebra, see Definition 2.1), and we reserve the term *irreducible* for the latter. Accordingly, a single operator T could be called transitive or irreducible if the algebra of polynomials in T has that property.

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