VANISHING COEFFICIENTS AND BINOMIAL CONGRUENCES

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Let K be any field of prime characteristic p, let $A(z) = \sum_{j=0}^{\infty} A_j z^j$ be any given polynomial over K of degree e (so that $A \neq 0$, write $A_j = 0$ for $j > e$), and, given a non-negative integer b, let

$$(1 + z)^b A(z) = E(z) = \sum_{j=0}^{\infty} E_j z^j$$

(where $E_j = 0$ for $j > b + e$). Also let n be any positive integer. Abhyankar ([1], [2, Lemmas 6.5 and 6.6]) recently obtained two somewhat curious results concerning this situation, which may be stated as follows:

Theorem 1 (Abhyankar). *If either* (i) $e < p^{n-1}$ *and* $E_1 = E_2 = \cdots = E_{p^{n-1}} = 0$ *or* (ii) $E_1 = E_2 = \cdots = E_{p^{n-1}+e} = 0$, *then* $b + e \equiv 0 \pmod{p^n}$.

However, the contributions of the alternative hypotheses (i), (ii) are not easy to trace in Abhyankar's proofs, and the question of how far these hypotheses might be separately weakened or simultaneously generalized was left open, though sharper results of this type might be desirable for the applications (in algebraic geometry) which he had in view. By approaching the problem from a quite different direction, we shall establish here the following more general result (which seems not to be accessible by Abhyankar's methods):

Theorem 2. *If* $E_1 = E_2 = \cdots = E_d = 0$, *where* $d = p^{n-1}(1 + [e/p^{n-1}])$, *then* $b + e \equiv 0 \pmod{p^n}$.

Here square brackets denote the usual "greatest integer part" function. Of course $d = p^{n-1}$ in case (i) above, while also $d \leq p^{n-1}(1 + (e/p^{n-1})) = p^{n-1} + e$ always, so Theorem 2 does indeed include both of Abhyankar's results as special cases. We note also that, since $1 + [x] > x$ for all real x, we always have $d > e$.

We shall establish Theorem 2 by first reducing it to the following property of binomial coefficients modulo p:

Theorem 3. *If* $\binom{-b}{e} \not\equiv 0 \pmod{p}$

$$\textit{and } \binom{-b}{j} \equiv 0 \pmod{p} \qquad (j = e + 1, \cdots, p^{n-1}(1 + [e/p^{n-1}])),$$

then $b + e \equiv 0 \pmod{p^n}$.

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