ANALYTIC SOLUTIONS OF THE HEAT EQUATION

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1. **Introduction.** In a recent paper [3] by P. C. Rosenbloom and D. V. Widder, expansions of solutions $u(x, t)$ of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1.1}$$

in series of the polynomial solutions $v_n(x, t)$ were considered. Explicitly

$$v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \frac{t^k}{k!}, \tag{1.2}$$

where $[n/2]$ means the greatest integer $\leq n/2$. In [3] criteria for expansions

$$u(x, t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} v_n(x, t), \qquad a_n = \frac{\partial^n}{\partial x^n} u(0, 0), \tag{1.3}$$

were obtained. In the present note we obtain a new form of necessary and sufficient condition on $u(x, t)$ for the validity of (1.3). We show that the expansion holds in some infinite strip $|t| < c$ if and only if (1.1) is satisfied and $u(x, t)$ is analytic, as a function of two variables, at some point of the x-axis. In fact if the point of analyticity is the origin, then (1.3) is a certain "diagonal series" taken from the Maclaurin double series of $u(x, t)$. We consider also the two simple series obtained by summing the Maclaurin series by rows or by columns. We show by examples that each of these simple series may converge at points outside the strip of convergence of the Maclaurin series.

We also derive a necessary and sufficient condition on a function $f(x)$ that it should represent temperatures on the x-axis, $u(x, 0) = f(x)$, deriving from earlier analytic temperatures. The width of the strip of analyticity depends on the growth properties of $f(x)$ as an entire function. The precise statement of this result is given in Corollary 3.1b below.

2. **Polynomial expansions.** The relation between polynomial series expansions of the form (1.3) and Maclaurin expansions of temperature functions is described in the following theorem.

THEOREM 2.1. *A solution $u(x, t)$ of* (1.1) *has an expansion* (1.3), *valid in the strip $|t| < c$, $-\infty < x < \infty$, if and only if it is equal to its Maclaurin expansion in the square $|t| < c$, $|x| < c$.*

Assume first that in the square in question $u_{xx} = u_t$ and that

$$u(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \frac{x^m}{m!} \frac{t^n}{n!}. \tag{2.1}$$

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