NORMALCY IN OPERATOR ALGEBRAS

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1. Introduction. In [9, Theorem 5; 393], von Neumann proves that each weakly (equivalently, strongly) closed self-adjoint algebra of operators on a Hilbert space (von Neumann algebra) which contains the identity operator I enjoys the double commutant property. If we denote the algebra of all those bounded operators commuting with a given family $\mathfrak{F}$ of operators by $\mathfrak{F}'$ (called the commutant of $\mathfrak{F}$), this result may be phrased as: $\mathcal{R} = \mathcal{R}''$, for von Neumann algebras $\mathcal{R}$ containing I. It is one of the key results of the theory.

This double commutant result expresses an algebraic property (called normalcy) of the algebra of all bounded operators. In the process of determining which properties of the algebra of bounded operators (factor of type I) are shared by the other types of factors (von Neumann algebras whose center consists of scalar multiples of I), Murray and von Neumann raise the question of which factors are normal [4; 183]. They construct factors of type II which are not normal [4, Lemma 3.4.2; 209] and conjecture that no factor of type II is normal [4; 185]. In [8, Lemma 4.4.2 (iii)], non-normal factors of type III are exhibited. It is shown that all factors of type II are non-normal in [2]. The question of normalcy for a factor can be phrased in terms of the fixed algebra under groups of unitarily induced (inner) automorphisms of the factor. We speculated on the possibility that the normalcy assertion phrased so as to allow *-automorphisms might be valid for all factors of type II_1. Singer disproved this in [10; 126]. A Galois theory result of this nature is established in [6], [7] for finite groups of outer automorphisms of finite factors.

A known and easy extension of von Neumann's result states that each von Neumann algebra of type I is normal—where one now tests only those von Neumann subalgebras satisfying the obvious necessary condition that they contain the center (see [1; 307, Exercise 13b], for example). We say that a von Neumann subalgebra $\mathcal{R}_0$ of a von Neumann algebra $\mathcal{R}$ is normal in $\mathcal{R}$ when it has the double commutant property relative to $\mathcal{R}$, i.e. $(\mathcal{R}_0' \cap \mathcal{R})' \cap \mathcal{R} = \mathcal{R}_0$. We shall prove that if $\mathcal{R}_0$ is of type I, it is normal in $\mathcal{R}$ if and only if its center is normal in $\mathcal{R}$ (cf. Theorem 1). Dealing, then, with Abelian von Neumann subalgebras of $\mathcal{R}$, we give some (easy) equivalent conditions to normalcy in $\mathcal{R}$ (cf. Theorem 2). We show that those Abelian von Neumann subalgebras which are totally-atomic over the center of $\mathcal{R}$ are normal in $\mathcal{R}$ (cf. Definition 3 and Theorem 4). In the last section, we give an example of an Abelian von Neumann subalgebra of a factor of type II_1 which is not normal in it.

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