GENERALIZED WEIGHTED m-th POWER PARTITIONS OVER A FINITE FIELD

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1. **Introduction, notation, and preliminaries.** Let $GF(q)$ denote the finite field of $q = p^n$ elements. For $\alpha \varepsilon GF(q)$ let

$$(1.1) \qquad e(\alpha) = e^{2\pi i t(\alpha)/p}, \quad \text{where} \quad t(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}}.$$

Then for positive integers t, k, m and $\alpha, \alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k \varepsilon GF(q)$ with $\alpha_1 \cdots \alpha_k \neq 0$, define

$$(1.2) \qquad \begin{aligned} S_m(\alpha, t, k) &= S_m(\alpha, t; \alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_k) \\ &= \sum e(\beta_1 \mid A_1 \mid + \cdots + \beta_k \mid A_k \mid), \end{aligned}$$

where the summation is over all sets $A_1, \cdots, A_k$ of $t \times t$ matrices over $GF(q)$ which satisfy the equation

$$(1.3) \qquad \alpha_1 \mid A_1 \mid^m + \cdots + \alpha_k \mid A_k \mid^m = \alpha,$$

where $\mid A \mid$ denotes the determinant of matrix A.

In this paper a number of results concerning the sum (1.2) are obtained. In §2 it is shown that $S_m(\alpha, t, k)$ can be expressed, by means of a reduction formula, in terms of certain $S_m(\alpha, 1, j)$ for $1 \leq j \leq k$. For $m = 1$, this leads (Theorem 1) to an explicit formula for $S_1(\alpha, t, k)$. In §3, $S_1(\alpha, t, k)$ is also found by a direct calculation, thus giving a check on the general reduction formula. The special case $m = 2r$, q even, is shown in §4 to reduce to the case $m = 1$ considered for the $2r$-th roots of α and the α_i. For $m = 2$ and q odd, the general reduction formula yields a formula (Theorem 2) for $S_2(\alpha, t, k)$ in terms of certain Kloosterman sums, which for $t = 1$ reduces to formulas given by Carlitz [2, §2]. In §6, a result of Carlitz and Uchiyama [5, §1] is used to obtain a bound (Theorem 3) on $S_m(\alpha, t, k)$ for m of arbitrary size such that $p \nmid m$.

We remark that for $\beta_i = 0$ all i, the sum (1.2) reduces to $N_m(\alpha, t, k) = N_m(\alpha, t; \alpha_1, \cdots, \alpha_k)$, the number of matrix solutions of equation (1.3). This special case of (1.2) will be considered for certain special values of m in a separate paper to appear soon. By using the reduction formula (2.4) with $\beta_i = 0$ all i and results concerning (1.3) for the case $t = 1$, which have been obtained in recent years by Carlitz, Corson, Faircloth, Hua, Vandiver, Weil and others (see [4] for a list of references), some results concerning (1.3) for arbitrary t will be given in this separate paper.

Except as indicated, lower case Greek letters, $\alpha, \beta, \gamma, \cdots$ will denote numbers of $GF(q)$. The exponential function defined by (1.1) has the properties that $e(\alpha + \beta) = e(\alpha)e(\beta)$ and

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