PARABOLIC EQUATIONS WITH NON-SMOOTH COEFFICIENTS

BY D. G. ARONSON

1. **Introduction.** It is known that the initial value problem for a linear system of parabolic differential equations of arbitrary order with variable coefficients has a unique regular solution provided that the coefficients of the system and the data satisfy certain rather mild smoothness and growth conditions (see, e.g., [1], [2]). In this note we investigate the nature of solutions of such initial value problems when the smoothness conditions on all but the highest order coefficients are relaxed. For this purpose we introduce a class $\mathcal{C}$ of generalized solutions of the initial value problem and show that under certain conditions the problem has a unique solution belonging to $\mathcal{C}$. Moreover, we obtain a sufficient condition for differentiability with respect to the space variables of solutions in $\mathcal{C}$. The definition of generalized solution which we employ here is motivated by the author's work [2] on the uniqueness of classical solutions of the initial value problem. Indeed, the main idea of the uniqueness proofs in [2] carries over to the present case.

In the major portion of this paper (§§2–7) we consider the second order linear parabolic differential operator

$$\mathcal{L} \equiv \Delta + \sum_{i=1}^{n} a_i(x, t)\, \partial/\partial x_i + a(x, t) - \partial/\partial t \tag{1.1}$$

defined for $(x, t) \in E^n \times [0, T] \equiv \bar{R}$, where Δ is the n-dimensional Laplace operator. The only assumption which we make concerning the coefficients a_i and a of $\mathcal{L}$ is that they belong to certain Banach spaces which will be defined in §2. The initial value problem for $\mathcal{L}$ with which we deal has the form

$$\mathcal{L}u = 0 \quad \text{for} \quad (x, t) \in E^n \times (0, T]; \qquad u(x, 0) = g(x) \quad \text{for} \quad x \in E^n, \tag{1.2}$$

where g is a given function. Actually our results extend to more general operators and to non-homogeneous systems of parabolic equations. In order to avoid unnecessary complications, we prefer to concentrate on the special case (1.1) which will serve to illustrate our methods, and defer a discussion of the generalizations until §8 (cf. Theorem IV, §8).

2. **Statement of results.** Let $I = (0, T]$ for some fixed $T > 0$ and $R = E^n \times I$. We say that $f(x, t) \in L_{\alpha,\beta}(R)$ for $\alpha, \beta \geq 1$ if

$$\| f \|_\alpha (t) \equiv \left\{ \int | f(x, t) |^\alpha \, dx \right\}^{1/\alpha} < \infty$$

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