AN ELEMENTARY METHOD IN THE ASYMPTOTIC THEORY OF NUMBERS

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1. **Introduction.** Let n denote a positive integral variable. In considering the asymptotic average of a (complex-valued) arithmetical function $f(n)$, it is often advantageous to express $f(n)$ as a Dirichlet product of functions $g(n)$, $h(n)$,

$$f(n) = \sum_{d\delta = n} g(d)h(\delta). \tag{1.1}$$

A noteworthy example of such a representation is the Dedekind-Liouville evaluation of the Euler totient function $\phi(n)$,

$$\phi(n) = \sum_{d\delta = n} \mu(d)\ \delta, \tag{1.2}$$

where $\mu(n)$ represents the Möbius function [5, Theorem 262]. The latter representation is fundamental in Mertens's classical treatment of the average order of $\phi(n)$, (cf. [5, Theorem 330] and (3.8) below).

Sometimes, however, a function $f(n)$ may not arise naturally in the form (1.1) and may therefore resist an elementary analysis of the Mertens type. To study certain functions of this category without appealing to the theory of generating functions, the author has used in two previous papers [3], [4] a "unitary" analogue of (1.1). We shall call $f(n)$ the unitary product of functions $g(n)$, $h(n)$ if $f(n)$ admits of the representation,

$$f(n) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} g(d)h(\delta). \tag{1.3}$$

A divisor d of n is called unitary if $d\delta = n$, $(d, \delta) = 1$, so that the summation in (1.3) is restricted to the unitary divisors d of n. An example of a function which admits of a simple representation of this form is the Dirichlet totient $\phi^*(n)$, defined to be the number of integers in a complete residue system (mod n) not divisible by a unitary factor d of n, other than $d = 1$. In particular, it was shown in [3, (2.8)] that

$$\phi^*(n) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} \mu^*(d)\ \delta, \tag{1.4}$$

where $\mu^*(n) = (-1)^{\omega(n)}$ and $\omega(n)$ denotes the number of (distinct) prime divisors of n. This representation was useful in obtaining an elementary estimate for the average order of $\phi^*(n)$, corresponding to the Mertens estimate for the Euler function ([3, Corollary 4.1.2], Corollary 3.1.3 below).

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