SOME FORMULAS OF JENSEN AND GOULD

By L. Carlitz

In a recent paper [1], Gould has discussed Jensen's formula [2]

$$\sum_{k=0}^{n} \binom{\alpha + \beta k}{k} \binom{\gamma - \beta k}{n - k} = \sum_{k=0}^{n} \binom{\alpha + \gamma - k}{n - k} \beta^k \tag{1}$$

from the point of view of "Vandermonde convolutions" and also proved the Abel-type analog

$$\sum_{k=0}^{n} \frac{(\alpha - \beta k)^k}{k!} \frac{(\gamma - \beta k)^{n-k}}{(n - k)!} = \sum_{k=0}^{n} \frac{(\alpha + \gamma)^k}{k!} \beta^{n-k}. \tag{2}$$

The question is raised whether other expansions of this sort can be found.

In the present note we show that

I. *If* $\{Q_k(x)\}$ *is a sequence of polynomials,* $Q_0(x) = 1$, deg $Q_k(x) = k$, *such that*

$$\sum_{k=0}^{n} Q_k(\alpha + \beta k) Q_{n-k}(\gamma - \beta k) = \sum_{k=0}^{n} \beta^k Q_{n-k}(\alpha + \gamma - k) \tag{3}$$

for all α, β, γ *then*

$$Q_n(\alpha) = \binom{\alpha}{n} \qquad (n = 0, 1, 2, \cdots). \tag{4}$$

II. *If* $\{Q_k(x)\}$ *is a sequence of polynomials,* $Q_0(x) = 1$, deg $Q_k(x) = k$, *such that*

$$\sum_{k=0}^{n} Q_k(\alpha + \beta k) Q_{n-k}(\gamma - \beta k) = \sum_{k=0}^{n} \beta^k Q_{n-k}(\alpha + \gamma) \tag{5}$$

for all α, β, γ *then*

$$Q_n(\alpha) = \frac{\alpha^n}{n!} \qquad (n = 0, 1, 2, \cdots). \tag{6}$$

Proof of I. We assume the truth of the theorem for $k = 0, 1, \cdots, n - 1$. Then (3) becomes

$$Q_n(\alpha + \beta n) + \sum_{k=1}^{n-1} \binom{\alpha + \beta k}{k} \binom{\gamma - \beta k}{n - k} + Q_n(\gamma) = Q_n(\alpha + \gamma) + \sum_{k=1}^{n} \beta^k \binom{\alpha + \gamma - k}{n - k}.$$

Using (1), this reduces to

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