# AXIOMS FOR LOCAL HOMOLOGY THEORY

By Thomas R. Brahana

**Introduction.** Local homology theory is concerned with certain homology groups which are associated with a point $x_0$ of a topological space $X$, defined in such a way that they are invariants of local character. It is the purpose of this paper to provide a set of axioms for local homology theory and an outline of the proof of a uniqueness theorem for any class of spaces containing the locally triangulable spaces. This theorem is analogous to the Uniqueness Theorem of Eilenberg and Steenrod, *Foundations of Algebraic Topology*, p. 100.

A few brief historical and heuristic remarks are now given. Local homology invariants have been defined and their properties examined in numerous articles and books. (See, for example, [10], [5], [2], [9; 121], [11], [14; 191], [1; 198], [12], [7], [8], [3], [4]. This listing is more or less chronological.) None of the above references contains all the paraphernalia of a "homology theory". In particular, the relative local homology groups and a description of the behavior of local homology groups under mappings do not usually appear (see, however, [12], [3] and [7]). On the other hand, it is more or less clear how one must proceed in each case in order to extend the definitions, so that a given local homology invariant leads to an associated "local homology theory". This procedure is carried out in §3 for an example, essentially the definition of [10]. (See also [9; 319, Note 20].)

The local homology invariants of the examples fall into two classes, depending on the values given to a point in $E^n$. The nature of the difference between these two classes can be seen by comparing the definitions in [10] and [9]. In the first, one considers as ($q$-dimensional) local homology group of a vertex in a complex the ($q$-dimensional) homology group of the closed star of the vertex modulo the boundary of the closed star of the vertex. In the second, one defines the ($q$-dimensional) local homology group to be the ($q$-dimensional) homology group of the boundary of the closed star of the vertex. Thus, the examples of the first class, (namely [10], [5], [2], [1], [14], [8; 351], [13]) assign the only non-trivial local homology group (or local Betti number) of $E^n$ to dimension $n$, while those of the second class (namely [9], [11] and [8; 357]) assign the only non-trivial local homology group to dimensions $n - 1$ and 0. The factor which determines which value is given is whether or not the special point itself is contained in the point sets which are used during the construction of homology invariants in the definition.

Any axiomatic account of local homology theory will give preference to one of these classes, first with regard to the manner the dimension indices are assigned, and secondly, in the way the "reduced local homology" is defined. The axioms

Received August 21, 1957; in revised form, February 28, 1958.