LOCAL OPERATORS ON FOURIER TRANSFORMS

By Louis de Branges

Let $f(x) = \int e^{ixt}F(t)\,dt$ be the Fourier transform of a function $F(x) \in L^1(-\infty, \infty)$. If $f(x)$ is a differentiable function of x, it may be possible to obtain the derivative $f'(x)$ by differentiating underneath the integral sign. The formula

$$f'(x) = \int e^{ixt}itF(t)\,dt \tag{1}$$

can be justified if the integral on the right converges absolutely. For

$$\frac{f(x+h) - f(x)}{h} = \int \frac{e^{iht} - 1}{iht}\, itF(t)\,dt$$

where

$$\left| \frac{e^{iht} - 1}{iht} \right| = \left| \frac{2}{ht} \sin \frac{ht}{2} \right| \leq 1,$$

and we can use the Lebesgue dominated convergence theorem.

We will take the formula (1) as our definition of a "derivative" on absolutely convergent Fourier transforms. From the point of view of (1), it is neater to drop the factor i. The operator H we get in this way corresponds to $-i$ times differentiation. An absolutely convergent Fourier transform $f(x) = \int e^{ixt}F(t)\,dt$ is to be in the domain of H and have image $H \cdot f(x) = g(x)$ if $g(x) = \int e^{ixt}tF(t)\,dt$ converges absolutely. We can now use the Lebesgue dominated convergence theorem to show that

$$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{ih}. \tag{2}$$

The formula (2) has interesting consequences from the point of view of the operational calculus.

Definition. An *operator* will be defined by means of a measurable complex-valued function $K(x)$ on the real line, and we denote the operator itself by $K(H)$. An absolutely convergent Fourier transform $f(x) = \int e^{ixt}F(t)\,dt$ is to be in the domain of $K(H)$, and the action of $f(x)$ is to be $g(x)$ if $g(x) = \int e^{ixt}K(t)\,F(t)\,dt$ is absolutely convergent.

When $K(x)$ is a polynomial in x, the corresponding operator $K(H)$ has a

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