# UNITARY DILATIONS OF OPERATORS

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1. **Introduction.** Let $A$ be an arbitrary operator on a Hilbert space $H$, and suppose $\| A \| \leq 1$ (such an operator is called a contraction). Then one may construct a second Hilbert space $K$, containing $H$ as a subspace, and a unitary operator $U$ on $K$, such that if $P$ is the projection of $K$ onto $H$, then

$$A^n x = PU^n x, \tag{1}$$

for all $n \geq 0$ and all $x \in H$ ([2], [6]); moreover, the space $K$ and the operator $U$ are determined uniquely within unitary equivalence by a certain natural minimality requirement. We shall say that $U$ is the *unitary dilation* of $A$ when it satisfies (1). Equation (1) was first established by Halmos [2] for the case $n = 1$, and later by Nagy [6] in the general case. An algebraic formulation, in terms of matrices with operator entries, has been given by Schaeffer [7]. In this paper we shall explicitly identify the unitary dilations of several types of operators; namely proper contractions (that is, the norm is strictly less than 1) and projections. Specifically, we prove that the unitary dilation of a proper contraction is unitarily equivalent to the $\alpha$-fold copy of the so-called bilateral shift operator, where $\alpha \leq \aleph_0$ is the dimension of $H$, and consequently that the unitary dilations of any two proper contractions on the same space are unitarily equivalent. Similar results are obtained for projections.

Technically speaking, this work is in close analogy with the measure-theoretic treatment of spectral theory for normal operators (see, for instance, [3]), and, in fact, leads to some results on functional representation for non-normal operators, which we shall present in a later note.

2. **Definitions.** We shall be dealing with a complex Hilbert space, of arbitrary dimension for the most part, and shall use the term "operator" to mean "bounded linear transformation." An operator $A$ is called a (*proper*) *contraction* if $\| A \| \leq 1$ ($\| A \| < 1$). The definitions of all terms employed without comment are to be found in [5]. The symbol $C$ will be reserved throughout for the perimeter of the unit circle in the complex plane. We write $m$ for Lebesgue measure on $C$, normalized so that $m(C) = 1$. The *support* $\Lambda(F)$ of a countably additive set function $F$ defined on the Borel sets of the complex plane is the complement of the union of all open sets on which $F$ vanishes. The spectrum of an operator $T$ will be denoted $Sp(T)$.

Definition 2.1. An *operator measure* on a Hilbert space $H$ is a function $F$ from Borel sets in the complex plane to positive operators on $H$ which is

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