NOTE ON A THEOREM BY A. BRAUER

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In a paper published some years ago in this journal (see [1], p. 78, Lemma 3) A. Brauer uses and proves the following property of Cassini's curves:

If for real a, b, c, k we have $a < c < k$ and $b \leq c$, then the set of points in the z-plane with

$$|z - b| \, |z - c| \leq (k - b)(k - c)$$

lies in the open domain

$$|z - b| \, |z - a| < (k - b)(k - a),$$

with the exception of the point $z = k$, which is the only common point of the boundaries of both sets.

We give in this note a new proof of this theorem, which is somewhat simpler than that given by A. Brauer.

Without loss of generality we can assume that $b = 0$, $k = 1$ and therefore $0 = b \leq c < k = 1$, $a < c$.

We denote the curve in the z-plane with the equation

$$|z| \, |z - a| = 1 - a \tag{1}$$

by $G(a)$. $G(a)$ consists of one or two Jordan curves and $G(0)$ is the unit-circle (see [2], pp. 137–140).

We have to prove that for $0 \leq c < 1$ the interior of $G(c)$ and $G(c)$ itself, save the point 1, are completely contained in the interior of $G(a)$ for $a < c$. We distinguish two cases:

Case i. $0 \leq a < c < 1$. We observe first that all points of $G(a)$ are contained in the interior of the unit-circle $G(0)$. Indeed, if we have in (1) $|z| \geq 1$, it follows that

$$1 - a \geq |z - a| \geq |z| - a \geq 1 - a,$$

then we have the equality sign everywhere in this relation and this is only possible for $z = 1$.

Suppose now that for $0 < a_0 < c_0 < 1$, $G(c_0)$ has a point z_0 which does not lie in the interior of $G(a_0)$. If z_0 does not lie on $G(a_0)$ itself, we let a_0 decrease continuously to 0. $G(a_0)$ tends then to the unit-circle which contains z_0 inside. Since $G(a_0)$ also varies continuously, z_0 lies for an intermediate value of a_0 on $G(a_0)$. Put $z_0 = re^{i\varphi}$ for z into the equation (1). Then this equation is satisfied both for $a = a_0$ and $a = c_0$. But for $z = z_0$ equation (1) becomes:

$$r^2(r^2 + a^2 - 2ar\cos\varphi) = (1 - a)^2,$$

$$F(a) \equiv (1 - r^2)a^2 - 2(1 - r^3\cos\varphi)a - r^4 + 1 = 0.$$